A short description of kinetic models for chemotaxis

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Abstract

We describe how the Keller-Segel model can be obtained as a drift-diffusion limit of kinetic models. Three different examples with global kinetic solutions illustrate yield different chemotactical sensitivity functions, including the case of a constant coefficient, where blow up in the limit may occur, the case with density threshold and an intermediate case for which the corresponding perturbed Keller-Segel models have global solutions.

1 Introduction

The amoeba *Dictyostelium discoideum* has a complex social behavior that has long attracted the attention of scientists from different fields. For mathematicians, the most widely used model is the Keller-Segel model (see [14, 15]; for a earlier version, see [20]). This model describes a population of cells moving toward higher concentrations of a certain chemical substances produced by themselves. It was derived from Fick's law, namely, by considering currents respectively for the cell and the chemo-attractant concentrations $\rho(x,t)$ and $S(x,t)$ defined in $(x,t)\in\Omega\times\mathbb{R}_+$, $\Omega\subset\mathbb{R}^2$, given by

$$\begin{aligned} J_\rho &:= \kappa_1\nabla S-\kappa_2\nabla\rho\ , \\ J_S &:= -\kappa_3\nabla S\ , \end{aligned}$$

and associated to the conservation of mass

$$\partial_t\phi=\nabla\cdot J_\phi+\mathcal{Q}_\phi\ ,$$

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where $\mathcal{Q}_\rho = 0$ and $\mathcal{Q}_S = \rho$ are the production/destruction terms for ρ and S. Considering certain normalizations and the limit of high diffusion (where the diffusion of the chemical substance is considered much higher that the diffusion of cells), the simplified version of the Keller-Segel model is given by:

$$\partial_t \rho = \nabla \cdot (\nabla \rho - \chi(S,\rho)\rho \nabla S) \ , \tag{1}$$

$$\delta \partial_t S - \Delta S = \rho \ , \tag{2}$$

$$\rho(\cdot, 0) = \rho^{\mathrm{I}} \ , \tag{3}$$

where $\rho(x,t) \geq 0$ and $S(x,t) \geq 0$ if $\rho^{\mathrm{I}} \geq 0$, $\delta \in \{0,1\}$, satisfy suitable decay conditions at infinity (or Neumann boundary conditions on the border, for bounded Ω). χ is called the chemotactical sensitivity.

These equations (and some of their generalizations) have attracted much attention. In particular, the exhibition of precise conditions such that their solutions exist globally or present finite-time-blow-up is an important mathematical question. In [13], for the case $\delta = 0$, it was proved the existence of values $C(\Omega)$ and $C^*(\Omega)$ such that, for the conserved total mass $M = \int_\Omega \rho^{\mathrm{I}} \mathrm{d}x$, if $\chi M < C(\Omega)$ or $\chi M > C^*(\Omega)$, then solutions exist globally or present finite-time-blow-up, respectively.

In [7] some of the blow up profiles were described as Dirac-delta type concentrations. In [16] it was proved that $C(\Omega) \geq 4\pi$ and, in particular, for radial solutions, $C(B_R) = 8\pi$, where $B_R \subset \mathbb{R}^2$ is a ball centered in the origin with arbitrary radius R. Later on, in [17], it was proved for general, but bounded Ω, that if there is a blow up point, then $\chi M \geq 8\pi$ if it is in the interior of Ω, or $8\pi > \chi M \geq 4\pi$ if the blow-up occurs on the border $\partial\Omega$.

For the whole space, with $\delta = 0$, the problem was solved in [5], where it was proved that $C(\mathbb{R}^2) = C^*(\mathbb{R}^2) = 8\pi$. For bounded domains, this problem is still open (see [6]). For more detailed reviews on chemotaxis, see also [10, 11].

The Keller-Segel model was first obtained from the phenomenological viewpoint. Its derivation from more basic principles was obtained in [21], as the limit dynamics of moderately interacting stochastic many-particle systems. A second approach was introduced in [19, 9] where the Keller-Segel model was formally obtained from kinetic models for chemotaxis, introduced in [1, 18]. The rigorous derivation of the Keller-Segel model from kinetic models was given in [3] for the case $\delta = 0$ and $\Omega = \mathbb{R}^3$ and were generalized in [12].

Here we consider only $\Omega = \mathbb{R}^2$, despite the fact that all theorems can be generalized (with minor modification) to the case $\Omega = \mathbb{R}^3$. For bounded domains, there are no results available. We also assume, for simplicity, $\delta = 0$, but similar results hold for $\delta = 1$. See [4, 12].

The kinetic models for chemotaxis consist in a transport equation for the phase space cell density, i.e., $f(x,v,t)$, where $v \in V$ is the cell velocity, for a compact and spherically symmetric set of all possible velocities $V \subset \mathbb{R}^2$. Given a turning kernel,

$T[S,\rho](x,v,v',t)$, the rate of changing from velocity v' to velocity v, in a space-time point $(x,t)\in\Omega\times V$ in the presence of chemical substance S and cell concentration ρ, we have

$$\partial_t f(x,v,t) + v\cdot\nabla f(x,v,t) = \mathcal{T}[S,\rho](f)(x,v,t)\ , \qquad (4)$$

where

$$\mathcal{T}[S,\rho](f)(x,v,t) := \int_V \left(T[S,\rho](x,v,v',t)f(x,v',t) - T[S,\rho](x,v',v,t)f(x,v,t)\right)\mathrm{d}v'\ .$$

This equation should be coupled with Equation (2), where

$$\rho(x,t) = \int_V f(x,v,t)\mathrm{d}v\ . \qquad (5)$$

Initial conditions are given by

$$f(x,v,0) = f^{\mathrm{I}}(x,v) \geq 0\ . \qquad (6)$$

We simplify our notation putting $f=f(x,v,t)$, $f'=f(x,v',t)$, $T[S,\rho]=T[S,\rho](x,v,v',t)$ and $T^*[S,\rho]=T[S,\rho](x,v',v,t)$.

2 Formal and rigorous convergence

Going go back to the Othmer-Dunbar-Alt model (4–6) and (2) and considering typical values for all the variables involved, we re-scale the problem in the new variables $\bar{x}=x/x_0$, $\bar{t}=t/t_0$, $\bar{v}=v/v_0$, $\bar{f}=f/f_0$, $\bar{S}=S/S_0$, $\bar{T}=T/T_0$, $\bar{\rho}=\rho/\rho_0$. Then, equations (4–6) and (2) are modified to (we simplify the notation, dropping all bars)

$$\begin{aligned}\partial_t f + \frac{v_0}{x_0/t_0}v\cdot\nabla f &= T_0 t_0 v_0^n \int_V \left(T[S,\rho]f' - T^*[S,\rho]f\right)\mathrm{d}v'\ , &(7)\\ \frac{\delta}{S_0}\partial_t S - \frac{t_0}{x_0^2}\Delta S &= \frac{t_0\rho_0}{S_0}\rho\ , &(8)\\ \rho &= \frac{f_0 v_0^n}{\rho_0}\int_V f\mathrm{d}v\ . &(9)\end{aligned}$$

Now, we consider that $t_0 = x_0^2$ (*diffusive scaling*) and the the microscopic typical velocity v_0 is much larger that the typical macroscopic velocity x_0/t_0, i.e.,

$$\varepsilon := \frac{x_0/t_0}{v_0} \ll 1\ .$$

We impose that the collisional term is very strong, actually, of order ε^{-2}. We finally assume some normalizations, impose $\delta = 0$, so that the system (7–9) becomes

$$\begin{aligned} \varepsilon^2 \partial_t f_\varepsilon + \varepsilon v \cdot \nabla f_\varepsilon &= \int_V \left(T_\varepsilon[S_\varepsilon, \rho_\varepsilon] f'_\varepsilon - T^*_\varepsilon[S_\varepsilon, f_\varepsilon] f_\varepsilon \right) \mathrm{d}v' && (10) \\ -\Delta S_\varepsilon &= \rho_\varepsilon \ , && (11) \\ \rho_\varepsilon &= \int_V f_\varepsilon \mathrm{d}v \ . && (12) \end{aligned}$$

We now look for the drift-diffusion limit of the above model. Namely, we look for a set of equation such that its solution is a good approximation, for small ε, of the functions ρ_ε and S_ε (which are the macroscopically relevant variables).

We start by considering the (formal) expansions

$$f_\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots \quad \text{and} \quad S_\varepsilon = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \cdots \tag{13}$$

We also assume that, for the turning kernel, the expansion

$$T_\varepsilon[S, \rho] = T_0[S, \rho] + \varepsilon T_1[S, \rho] + \cdots \tag{14}$$

is well defined and that

(A0) There exists a bounded velocity distribution $F(v) > 0$, independent of x, t, and S, such that the detailed balance principle $T^*_0[S]F = T_0[S]F'$ holds. The flow produced by this equilibrium distribution vanishes, and F is normalized:

$$\int_V vF(v)\,\mathrm{d}v = 0\,, \qquad \int_V F(v)\,\mathrm{d}v = 1\,. \tag{15}$$

The turning rate $T_0[S]$ is bounded, and there exists a constant $\gamma > 0$ such that $T_0[S]/F \geq \gamma$, $\forall\,(v, v') \in V \times V$, $x \in \mathbb{R}^3$, $t > 0$.

We put expansions (13) and (14) in the system (10–12) and match terms of the same order of ε.

In order ε^0, we find

$$\begin{aligned} 0 &= \int_V \left(T_0[S_0, \rho_0] f_0 - T^*_0[S_0, \rho_0] f'_0 \right) \mathrm{d}v' \ , && (16) \\ -\Delta S_0 &= \rho_0 := \int_V f_0 \mathrm{d}v \ . && (17) \end{aligned}$$

From the equation

$$\iint_{V\times V} \left(T_0[S_0, \rho_0] f'_0 - T^*_0[S_0, \rho_0] f_0 \right) \frac{f_0}{F} \mathrm{d}v' \mathrm{d}v = \frac{1}{2} \iint_{V\times V} T_0[S_0, \rho_0] F' \left(\frac{f_0}{F} - \frac{f'_0}{F'} \right)^2 \mathrm{d}v \mathrm{d}v'$$

we deduce that the solution of Equation (16–17) is given by $f_0(x,v,t) = \rho_0(x,t)F(v)$ and $S_0(x,t) = (2\pi)^{-1}\int_V \log|x-y|\rho_0(y,t)\mathrm{d}y$, where $\rho_0(x,t)$ is the unknown macroscopic density.

Now, we go back to the System (10–12) and isolate terms of order ε^1

$$\begin{aligned} v\cdot\nabla f_0 &= \int_V \left(\mathcal{T}_1[S_0,\rho_0](f_0) + \mathcal{T}_0[S_0,\rho_0](f_1)\right)\mathrm{d}v' \;, && (18)\\ -\Delta S_1 &= \rho_1 := \int_V f_1 \mathrm{d}v \;. && (19)\end{aligned}$$

with

$$\mathcal{T}_k[S,\rho](f) := \int_V \left(T_k[S,\rho]f' - T_k^*[S,\rho]f\right)\mathrm{d}v' \;, k=0,1 \;. \tag{20}$$

Equation (18) can be solved with help of Lemma 2 of [3] so that

$$f_1(x,v,t) = -\kappa[S_0,\rho_0](x,v,t)\cdot\nabla\rho_0(x,t) - \Theta[S_0,\rho_0](x,v,t)\rho_0(x,t) + \rho_1(x,t)F(v)\;, \tag{21}$$

where κ and Θ are the solutions of

$$\begin{aligned} \mathcal{T}_0[S_0,\rho_0](\kappa) &= -vF\;, && (22)\\ \mathcal{T}_0[S_0,\rho_0](\Theta) &= \mathcal{T}_1[S_0,\rho_0](F)\;, && (23)\end{aligned}$$

and ρ_1, the macroscopic density of f_1, is a new unknown.

Back to the equations (10–12), after integrating order ε^2 terms, and using the previous results we find

$$\partial_t\rho_0 + \nabla\cdot\left(D(S_0,\rho_0)\nabla\rho_0 - \Gamma(S_0,\rho_0)\rho_0\right) = 0\;, \tag{24}$$

where the diffusivity tensor and the convection vector are given by

$$\begin{aligned} D[S_0,\rho_0](x,t) &= \int_V v\otimes\kappa[S_0,\rho_0](x,v,t)\mathrm{d}v\;, && (25)\\ \Gamma[S_0,\rho_0](x,t) &= -\int_V v\Theta[S_0,\rho_0](x,v,t)\mathrm{d}v\;. && (26)\end{aligned}$$

To finish the formal deduction, we only need to couple equation (24) to

$$-\Delta S_0 = \rho_0\;. \tag{27}$$

Let us define the symmetric and anti-symmetric parts of $T_\varepsilon[S,\rho]F$, respectively, by:

$$\begin{aligned} \phi_\varepsilon^S[S,\rho] &:= \frac{T_\varepsilon[S,\rho]F' + T_\varepsilon^*[S,\rho]F}{2}\,, && (28)\\ \phi_\varepsilon^A[S,\rho] &:= \frac{T_\varepsilon[S,\rho]F' - T_\varepsilon^*[S,\rho]F}{2}\,. && (29)\end{aligned}$$

Now, we are ready to state the *rigorous convergence results.* We will not prove them here, but the proofs can be found in references [3, 4, 12]

Theorem 1. *Let $F \in L^\infty(V)$ be a positive velocity distribution satisfying Assumption (A0) and let $\phi_\varepsilon^S[S]$ and $\phi_\varepsilon^A[S]$ be defined as in (28) and (29). Assume that there exist $q > 3$, $\lambda_0 > 0$, and a non-decreasing function $\Lambda \in L^\infty_{\mathrm{loc}}([0,\infty))$, such that*

$$\frac{f^{\mathrm{I}}}{F} \in \mathcal{X}_q := L^1_+ \cap L^q\left(\mathbb{R}^2 \times V;\; F\,\mathrm{d}x\,\mathrm{d}v\right)\,, \tag{30}$$

$$\phi_\varepsilon^S[S,\rho] \geq \lambda_0(1 - \varepsilon\Lambda(\|S\|_{W^{1,\infty}(\mathbb{R}^2)}))FF'\,, \tag{31}$$

$$\int_V \frac{\phi_\varepsilon^A[S,\rho]^2}{F\phi_\varepsilon^S[S,\rho]}\mathrm{d}v' \leq \varepsilon^2\Lambda(\|S\|_{W^{1,\infty}(\mathbb{R}^2)})\,. \tag{32}$$

Then there exists $t^* > 0$, independent of ε, such that the existence time of the local mild solution of (10–12) is larger than t^*, and the solution satisfies, uniformly in ε,

$$\begin{aligned} \frac{f_\varepsilon}{F} &\in L^\infty(0,t^*;\; \mathcal{X}_q)\,, \\ S_\varepsilon &\in L^\infty(0,t^*;\; L^p \cap C^{1,\alpha}(\mathbb{R}^2))\,, \quad \alpha < \frac{q-2}{q}\,,\ 3 < p < \infty \\ r_\varepsilon = \frac{f_\varepsilon - \rho_\varepsilon F}{\varepsilon} &\in L^2\left(\mathbb{R}^2 \times V \times (0,t^*);\; \frac{\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t}{F}\right)\,. \end{aligned} \tag{33}$$

Theorem 2. *Let the assumptions of Theorem 1 hold. Assume further that for any functions Σ_ε uniformly bounded (as $\varepsilon \to 0$) in $L^\infty_{\mathrm{loc}}(0,\infty;\; C^{1,\alpha}(\mathbb{R}^2))$ for some $0 < \alpha \leq 1$, such that Σ_ε and $\nabla\Sigma_\varepsilon$ converge to Σ_0 and $\nabla\Sigma_0$, respectively, in $L^p_{\mathrm{loc}}(\mathbb{R}^2 \times [0,\infty))$ for some $p > 3/2$ and η_ε converges to η_0 in $L^2_{\mathrm{loc}}(\mathbb{R}^2 \times [0,\infty))$, we have the convergence*

$$\begin{aligned} &T_\varepsilon[\Sigma_\varepsilon,\eta_\varepsilon] \to T_0[\Sigma_0,\eta_0] \qquad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^2 \times V \times V \times [0,\infty))\,, \\ &\frac{\mathcal{T}_\varepsilon[\Sigma_\varepsilon,\eta_\varepsilon](F)}{\varepsilon} = \frac{2}{\varepsilon}\int_V \phi_\varepsilon^A[\Sigma_\varepsilon,\eta_\varepsilon]\mathrm{d}v' \to \mathcal{T}_1[\Sigma_0,\eta_0](F) \qquad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^2 \times V \times [0,\infty))\,. \end{aligned}$$

Then, the solutions of (10–12) satisfy (possibly after extracting subsequences)

$$\begin{aligned} \rho_\varepsilon &\to \rho_0 \qquad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^2 \times (0,t^*))\,, \\ S_\varepsilon &\to S_0 \qquad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^2 \times (0,t^*))\,,\ 1 \leq q < \infty\,, \\ \nabla S_\varepsilon &\to \nabla S_0 \qquad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^2 \times (0,t^*))\,,\ 1 \leq q < \infty\,, \end{aligned}$$

and their limits are weak solutions of (24–27) subject to the initial condition

$$\begin{aligned} \rho_0(x,0) &= \int_V f^{\mathrm{I}}(x,v)\,\mathrm{d}v\,, \\ S_0(x,0) &= S^{\mathrm{I}}(x)\,. \end{aligned}$$

3 Models with global existence and their drift-diffusion limits

In this section, we give some particular examples of turning kernels for which it is possible to prove global existence of solutions. In some of these cases, it will be also possible to conclude bounds for the solution of the limit Keller-Segel models.

In this section we fix ε. We always assume $(A0)$. It is easy to see that all turning kernels obey the assumptions in Theorems 1 and 2. For Example 2, see Remark 1.

Example 1. *[12] Let us suppose that there is a constant C such that*

$$T_\varepsilon[S,\rho](x,v,v',t) \le C\left(1 + S(x+\varepsilon v,t) + S(x-\varepsilon v,t)\right) \ .$$

Assume further that $f^{\mathrm{I}} \in L^1_+ \cap L^\infty(\mathbb{R}^2 \times V)$ Then, there is a global solution $f(\cdot,\cdot,\cdot) \in L^1_+ \cap L^\infty(\mathbb{R}^2 \times V)$ and $S(\cdot,t) \in L^p(\mathbb{R}^2)$, $p \in [2,\infty]$, $\forall t \in [0,\infty)$ of the system (10–12) for any fixed $\varepsilon > 0$.

If, in the previous example, we assume that $T_\varepsilon[S,\rho](x,v,v',t) = \psi(S(x,t), S(x+\varepsilon v,t))$, for a smooth function $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ then the coefficients in the limit equation are given by (see [3])

$$\begin{aligned} D[S_0,\rho_0] &= \frac{1}{2\psi(S_0,S_0)}\int_V v^2 F(|v|)\mathrm{d}v \ , \\ \Gamma[S_0,\rho_0] &= \frac{\partial_2\psi(S_0,S_0)}{2\psi(S_0,S_0)}\int_V v^2 F(|v|)\mathrm{d}v \nabla S_0 \ , \end{aligned}$$

where ∂_2 denotes differentiation with respect to the second variable. If ψ is at most linear in the second variable, then global exitence is guaranteed. If, moreover, $\psi(S,\tilde{S}) = \Psi(\tilde{S}-S) \le A\tilde{S}+B$ for positive constants A and B, then we have as limit model the Keller-Segel model with constant coeficients D and χ, which presents finite-time-blow-up for certain initial conditions.

Now, we consider a two-parameters turning kernel depending on the phase-space density:

$$T_{\varepsilon,\mu}[S,f](x,v,v',t) = \Phi(S(x+\varepsilon\zeta_\mu(f(x,v,t))v,t) - S(x,t))F(v) \ , \tag{34}$$

where

$$C_0(\mu) := \sup_{y\ge 0}\zeta_\mu(y)y \ , \quad \lim_{\mu\to 0} C_0(\mu) = \infty \ , \tag{35}$$

and for increasing function Φ such that $0 < \Phi_{\min} \le \Phi(y) \le Ay + B$, A and B positive constants and $\zeta_\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and bounded for all $\mu \ge 0$.

Example 2. *[2] For any fixed $\mu \geq 0$ and $\varepsilon > 0$ there exist global solutions of the kinetic model (10–12) i.e., for any $t > 0$, $f_{\varepsilon,\mu} \in L^\infty(0,t;L^\infty(\mathbb{R}^2 \times V))$ and $S_{\varepsilon,\mu} \in L^\infty(0,t;L^\infty(\mathbb{R}^2))$. Furthermore, $||\rho_{\varepsilon,\mu}(\cdot,t)||_{L^\infty(\mathbb{R}^2)}$ and $||S_{\varepsilon,\mu}(\cdot,t)||_{L^\infty(\mathbb{R}^2)}$ are bounded by μ-independent functions. For strictly positive μ we find as the drift-diffusion limit of this model the perturbation of the Keller-Segel model introduced in [22, 23], i.e., with constant diffusivity and sensitivity given by*

$$\chi(S,\rho) = \frac{\Psi'(0)}{\Psi(0)} \int_V \zeta_\mu(\rho F(v))F(v)v^2 \mathrm{d}v \ . \tag{36}$$

Remark 1. *In order to extend Theorems 1 and 2 to the case where the turning kernel depends also on f, it is important to prove the convergence $||f_\varepsilon - f_0||_{L^p(\mathbb{R}^2\times V)} \to 0$, for some $p \in [1,\infty]$. This is a simple consequence of the convergence $\rho_\varepsilon \to \rho_0$ in $L^2(\mathbb{R}^2)$ and the boundedness of the remainder*

$$r_\varepsilon := \frac{f_\varepsilon - \rho_\varepsilon F}{\varepsilon} \ .$$

For details, see [2]. It is important to note, that, for $\mu > 0$, it is possible to prove global convergence, i.e., the maximum time t^* in Theorem 2 can be arbitrarily extended.

In the first example the turning kernel depended only of S, while in the second case we introduced a dependence on the cell density such that for high concentrations the turning kernel becomes constant. Now, we consider a stronger assumption such that the chemotactical part $T_\varepsilon - T_0$ vanishes for densities above a certain strictly positive threshold $\bar{\rho}$.

Example 3. *[4] Let us consider the turning kernel given by*

$$T_\varepsilon[S,\rho](x,v,v',t) = \Psi(S(x+\varepsilon\zeta(\rho)v,t) - S(x,t))F(v)$$

such that there is an upper bound $\bar{\rho}$, i.e., $\zeta(\rho) = 0$ for $\rho \geq \bar{\rho}$ or let us consider a turning kernel

$$T_\varepsilon[S,\rho](x,v,v',t) = \lambda[S,\rho](x,t)F(v) + \varepsilon F(v)a(S,\rho)v\cdot\nabla S$$

for ε small enough and $a(S,\rho) = 0$ for $\rho \geq \bar{\rho}$. We also impose that

$$\sup_{\rho\geq 0, S\geq 0} \frac{a(S,\rho)}{\rho - \bar{\rho}} < \infty \quad \text{and} \quad \sup_{\rho\geq 0, S\geq 0} \frac{\zeta(\rho)}{\rho - \bar{\rho}} < \infty \ .$$

Let us suppose that initial conditions are given by $f^{\mathrm{I}}(x,v) = \rho^{\mathrm{I}}(x)F(v)$, $\rho^{\mathrm{I}} \in L^1_+ \cap L^\infty(\mathbb{R}^2)$, $S^{\mathrm{I}} = 0$. Then the solution (f,S) of the nonlinear system (10–11) exists globally: $f \in L^\infty(0,\infty;L^1_+ \cap L^\infty(\mathbb{R}^2 \times V))$, $S \in L^\infty(0,t;L^p(\mathbb{R}^2))$, $p \in (1,\infty]$, $\forall t \in (0,\infty)$. Furthermore,

$$||\rho(\cdot,t)||_{L^\infty(\mathbb{R}^2)} \leq \left|\left|\frac{f(\cdot,\cdot,t)}{F}\right|\right|_{L^\infty(\mathbb{R}^2\times V)} \leq \max\{||\rho^{\mathrm{I}}||_{L^\infty(\mathbb{R}^2)}, \bar{\rho}\} \ , \ \forall t \in \mathbb{R}_+ \ .$$

As a direct consequence of the previous result (more specifically from the fact that the bound for $||\rho(\cdot,t)||_{L^\infty(\mathbb{R}^2)}$ is ε-independent), we reproduce the results in [8] (with some technical differences), namely the global existence of solution of Keller-Segel models with constant diffusivity and sensitivity such that $\chi(\rho) = 0$ for $\rho \geq \bar{\rho}$. For details, see [4].

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