On instability of equilibrium figures of rotating viscous incompressible liquid

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Abstract. We prove that non-axially symmetric equilibrium figures of uniformly rotating viscous incompressible liquid are unstable when the second variation of the energy functional can take negative values.

1. Introduction

Equilibrium figure $\mathcal{F}$ of an incompressible liquid subjected to the capillary and self-gravitation forces and rotating as a rigid body with the angular velocity ω about the x_3-axis is defined by the equation

$$\sigma\mathcal{H}(x)+\frac{\omega^2}{2}|x'|^2+\kappa\mathcal{U}(x)+p_0=0,\qquad x\in\mathcal{G}=\partial\mathcal{F},\tag{1.1}$$

where $\sigma=\mathrm{const}>0$ is the coefficient of the surface tension, $\mathcal{H}(x)$ is twice the mean curvature of the surface $\mathcal{G}$ at the point x negative for convex domains, $p_0=const$, $\mathcal{U}(x)=\int_{\mathcal{F}}|x-y|^{-1}dy$ is the Newtonian potential, $x'=(x_1,x_2,0)$ and κ is the gravitational constant. The case of the absence of self -gravitation ($\kappa=0$) is not excluded. The density of the liquid equals one. The velocity vector field and the pressure of the rotating liquid are given by

$$\mathcal{V}(x)=\omega(e_3\times x),\qquad \mathcal{P}(x)=\frac{\omega^2}{2}|x'|^2+p_0,$$

where $e_3=(0,0,1)$ is a unit vector in the direction of the x_3-axis. We assume that the equilibrium figure $\mathcal{F}$ is a bounded domain with a smooth boundary $\mathcal{G}$ and with the barycenter located at the origin which means that

$$\int_{\mathcal{F}}x_i dx=0,\qquad i=1,2,3.$$

The angular momentum of the rotating liquid, $\int_{\mathcal{F}}x\times\mathcal{V}(x)dx$, is parallel to the axis of rotation, i.e.

$$\int_{\mathcal{F}}x\times\mathcal{V}(x)dx=\beta e_3,$$

$$\int_{\mathcal{F}}x_jx_3dx=0,\quad j=1,2,\qquad \beta=\omega\int_{\mathcal{F}}|x'|^2dx.$$

In the present paper we continue the analysis of stability of equilibrium figures carried out in [1-5]. As in [2], we assume that $\mathcal{F}$ does not possess the property of axial symmetry with respect to the x_3-axis. In this case equation (1.1) defines a one-parameter family of equilibrium figures, $\mathcal{F}_\theta$, obtained by rotation of the angle θ of one of them, $\mathcal{F}_0$, about the x_3-axis. It is natural to assume that $\theta\in R$ and $\mathcal{F}_{\theta+2\pi}=\mathcal{F}_\theta$.

Given the family $\mathcal{F}_\theta$, we consider evolution free boundary problem for the perturbations of the velocity and of the pressure written in the coordinate system rotating about the x_3 -axis with the angular velocity ω. It consists in the determination of a bounded domain $\Omega_t \in R^3$, $t > 0$, of a vector field $v(x,t) = (v_1, v_2, v_3)$ and of a function $p(x,t)$, $x \in \Omega_t$, satisfying the relations

$$v_t + (v \cdot \nabla)v + 2\omega(e_3 \times v) - \nu\nabla^2 v + \nabla p = 0,$$

$$\nabla \cdot v = 0, \qquad x \in \Omega_t, \qquad t > 0, \tag{1.2}$$

$$T(v,p)n = (\sigma H(x) + \frac{\omega^2}{2}|x'|^2 + p_0 + \kappa U(x,t))n, \qquad V_n = v \cdot n, \qquad x \in \Gamma_t$$

$$v(x,0) = v_0(x), \qquad x \in \Omega_0.$$

Here ν is a constant positive viscosity coefficient, n is the exterior normal to the free surface $\Gamma_t = \partial\Omega_t$, V_n is the velocity of evolution of Γ_t in the normal direction, H is twice the mean curvature of Γ_t,

$$U(x,t) = \int_{\Omega_t} \frac{dy}{|x-y|}$$

is the Newtonian potential computed for the unknown domain Ω_t and, finally,

$$T(v,p) = -pI + \nu S(v)$$

and

$$S(v) = \nabla v + (\nabla v)^T = \Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)_{i,j=1,2,3}$$

are the stress and the doubled rate-of-strain tensors, respectively. The domain Ω_0 is given.

Concerning the initial data we assume that v_0 is a small divergence free vector field satisfying the compatibility conditions

$$S(v_0)n - n(n \cdot S(v_0)n)\Big|_{\Gamma_0} = 0$$

and Ω_0 is close to $\mathcal{F}_0$ which means that the surface Γ_0 can be given by the equation

$$x = y + N(y)\rho_0(y), \qquad y \in \mathcal{G} \tag{1.3}$$

with a certain small function $\rho_0(y)$; moreover, the total and angular momenta corresponding to $v_0 + \mathcal{V}$ are the same as for $\mathcal{V}$, i.e.

$$\int_{\Omega_0} v_0(x)dx = 0, \qquad \int_{\Omega_0} x \times (v_0(x) + \mathcal{V}(x))dx = \beta e_3. \tag{1.4}$$

It can be verified that this implies

$$\int_{\Omega_t} v(x,t)dx = 0, \qquad \int_{\Omega_t} x \times (v(x,t) + \mathcal{V}(x))dx = \beta e_3, \qquad \forall t > 0,$$

i.e.

$$\int_{\Omega_t} v(x,t)dx = 0,$$

$$\int_{\Omega_t} v(x,t)\cdot\eta_i(x)dx + \omega\int_{\Omega_t}\eta_3(x)\cdot\eta_i(x)dx = \omega\int_{\mathcal{F}}\eta_3(x)\eta_i(x)dx, \qquad i=1,2,3, \tag{1.5}$$

where $\eta_i(x) = e_i \times x$ is a vector of rigid rotation about the x_i- axis. Finally, for arbitrary $t \geq 0$ the conditions

$$|\Omega_t| = |\mathcal{F}|, \qquad \int_{\Omega_t} x_i dx = 0, \qquad i = 1,2,3. \tag{1.6}$$

are satisfied.

Problem (1.2) has a one-parameter family of stationary solutions $v(x,t) = 0,\ p(x,t) = 0,\ x \in \mathcal{F}_\theta,\ \theta \in [0,2\pi)$. The stability of these solutions depends on the properties of the second variation of the energy functional,

$$\delta^2 R[\rho] = \int_{\mathcal{G}}(\sigma|\nabla_{\mathcal{G}}\rho(y)|^2 - \beta(y)\rho^2(y))dS$$

$$+\frac{\omega^2}{\int_{\mathcal{F}}|x'|^2dx}\Big(\int_{\mathcal{G}}\rho(y)|y'|^2dS\Big)^2 - \kappa\int_{\mathcal{G}}\int_{\mathcal{G}}\rho(y)\rho(z)\frac{dS_ydS_z}{|y-z|}, \tag{1.7}$$

where

$$\beta(y) = \sigma(\mathcal{H}^2(y) - 2\mathcal{K}(y)) + \frac{\omega^2}{2}\frac{\partial}{\partial N}|y'|^2 + \kappa\frac{\partial\mathcal{U}(y)}{\partial N}, \tag{1.8}$$

and $\mathcal{K}$ is the Gaussian curvature of $\mathcal{G}$. In the paper [2] it was shown that if the quadratic form $\delta_0^2 R[\rho]$ considered on the set of functions satisfying the constrains

$$\int_{\mathcal{G}}\rho(y)dS = 0, \quad \int_{\mathcal{G}}\rho(y)y_i dS = 0, \quad i = 1,2,3, \tag{1.9}$$

$$\int_{\mathcal{G}}\rho(y)N(y)\cdot\eta_3(y)dS = 0 \tag{1.10}$$

is positive definite, then the regime of the rigid rotation is stable. This means that problem (1.2) with initial data (v_0, Ω_0) sufficiently close to $(0, \mathcal{F}_0)$ has a unique solution defined for all $t > 0$ and, as $t \to \infty$, $v(x,t) \to 0$ and $\Gamma_t \to \mathcal{G}_\varphi$ with a certain "asymptotic phase" φ. In the present paper we prove that it is not the case, if $\delta_0^2 R[\rho]$ can take negative values on the set (1.9). We assume that Γ_t can be prescribed by the equation of the type (1.3), namely,

$$x = \xi + \widehat{N}(\xi)\widehat{\rho}(\xi,t), \qquad \xi \in \mathcal{G}_{\vartheta(t)}, \tag{1.11}$$

where $\widehat{N}(\xi)$ is an exterior normal to $\mathcal{G}_{\vartheta(t)}$, and we interpret (1.2) as an initial-boundary value problem for v, p and $\widehat{\rho}$. We show that if (1.2) has a solution defined for all $t > 0$ with a small $v(x,t)$ and with Γ_t sufficiently close to $\cup_\theta \mathcal{G}_\theta$, then the function $\vartheta(t)$ can be defined in such a way that the functional

$$I(\vartheta(t)) = \int_{\mathcal{G}_{\vartheta}(t)} \widetilde{\rho}^2(\xi,t)dS \tag{1.12}$$

takes the minimal value among all similar functionals $I(\theta) = \int_{\mathcal{G}_\theta} \rho_\theta^2(y,t)dS$ such that

$$\Gamma_t = \{x = y + N_\theta(y)\rho_\theta(y,t), \quad y \in \mathcal{G}_\theta\}$$

where $N_\theta(y)$ is an exterior normal to $\mathcal{G}_\theta$. Then, analyzing the corresponding linearized problem we show that the solution of (1.2), $(v(x,t), \Gamma_t)$, can not stay always in a certain neighborhood of $(0, \mathcal{G}_{\vartheta(t)})$, which means the lack of stability. The exact formulation of the result is given below, in Sect. 4.

2. Auxiliary propositions

This section is devoted to calculations aimed at the determination of the function $\vartheta(t)$ (they are close to the arguments in [2], Sect.3). We recall some auxiliary constructions from [2]. It is well known that for every point $x \in R^3$ with $\mathrm{dist}(x, \mathcal{G}) \leq \delta_1$ where $\mathcal{G} \equiv \mathcal{G}_0$, $\delta_1 \ll 1$, the relation

$$x = y + N(y)r \qquad y \in \mathcal{G}, \tag{2.1}$$

with $|r| \leq \delta_1$ holds. Let us consider this relation more closely. Assume that $y \in G \subset \mathcal{G}$ where G is a subset of $\mathcal{G}$ given by

$$y = y(s), \qquad s = (s_1, s_2) \in \omega \subset R^2$$

(s_1, s_2 are local coordinates on G). The transformation

$$E(s_1, s_2, r) \;=\; y(s_1, s_2) + N(s_1, s_2)r \;\equiv\; y(s) + N(s)r$$

makes the set $U = \{s \in \omega;\ |r| \leq \delta_1\}$ correspond to the set V of the points (2.1) with $y \in G$, $|\rho| \leq \delta_1$.

Let $\mathcal{J}$ be the Jacobi matrix of $E(s_1, s_2, r)$, i.e.

$$\mathcal{J} = \begin{pmatrix} y_{1,s_1}(s) + N_{1,s_1}(s)\,r\,, & y_{1,s_2}(s) + N_{1,s_2}(s)\,r\,, & N_1(s) \\ y_{2,s_1}(s) + N_{2,s_1}(s)\,r\,, & y_{2,s_2}(s) + N_{2,s_2}(s)\,r\,, & N_2(s) \\ y_{3,s_1}(s) + N_{3,s_1}(s)\,r\,, & y_{3,s_2}(s) + N_{3,s_2}(s)\,r\,, & N_3(s) \end{pmatrix} \tag{2.2}$$

where $N_i(s) = N_i(y(s))$, $y_{k,s_j} = \frac{\partial y_k(s)}{\partial s_j}$, $N_{k,s_j} = \frac{\partial N_k(s)}{\partial s_j}$. The vectors $y_{,s_j} = (y_{k,s_j})_{k=1,2,3} \equiv \tau_j$, $j = 1,2$, are linearly independent and tangential to $\mathcal{G}$, hence, $\det \mathcal{J}\big|_{r=0} \neq 0$ and $\det \mathcal{J}(s,r) \neq 0$, since δ_1 is small. Therefore there exists the inverse transformation

$$E^{-1}(x) \;=\; \Big\{s = \Sigma(x),\ r = R(x)\Big\},$$

so that $U = E^{-1}V$. We denote by J_{km} the elements of $\mathcal{J}$ and by J^{km} the elements of $\mathcal{J}^{-1}$. It is clear that

$$x_{m,s_\alpha} \equiv \frac{\partial x_m}{\partial s_\alpha} = J_{m\alpha}, \quad \frac{\partial x_m}{\partial r} = J_{m3}, \quad \frac{\partial \Sigma_\alpha}{\partial x_k} = J^{\alpha k}, \quad \frac{\partial R}{\partial x_k} = J^{3k},$$

where $\alpha = 1,2,\ k = 1,2,3$. The elements J^{3k} are components of the vector

$$\frac{\vec{x}_{,s_1} \times \vec{x}_{,s_2}}{\det \mathcal{J}} = \frac{\vec{x}_{,s_1} \times \vec{x}_{,s_2}}{\vec{N} \cdot (\vec{x}_{,s_1} \times \vec{x}_{,s_2})}.$$

Since the surface $\mathcal{G}$ and the parallel surface $\mathcal{G}^{(r)}:\ \{x = y + N(y)r,\ y \in \mathcal{G}\}$ have a common normal $\vec{N}(y)$, and $\vec{x}_{,s_j}$ are linearly independent tangential vectors to $\mathcal{G}^{(r)}$, there holds

$$\frac{\vec{x}_{,s_1} \times \vec{x}_{,s_2}}{\det \mathcal{J}} = \frac{\vec{N}|\vec{x}_{,s_1} \times \vec{x}_{,s_2}|}{|\vec{x}_{,s_1} \times \vec{x}_{,s_2}|} = \vec{N}, \tag{2.3}$$

if the triple of vectors $\vec{y}_{,s_1}, \vec{y}_{,s_2}, \vec{N}$ has a right orientation. Hence, R is a function defined in δ_1-neighborhood of $\mathcal{G}$, and

$$\frac{\partial R}{\partial x_k} = J^{3k} = N_k(y)$$

(this follows also from the fact that $R(x) = \mathrm{dist}(x, \mathcal{G})$).

Now, let Γ be a closed surface located in the $\delta_1/2$- neighborhood of $\mathcal{G} = \mathcal{G}_0$. As a consequence, it can be prescribed by equation of the type (1.7), i.e.

$$x = y + N(y)\rho(y), \quad y \in \mathcal{G}_0 \equiv \mathcal{G}, \tag{2.4}$$

where $N = N_0$ and $|\rho(y)| \leq \delta_1/2$. We also consider the surface $\Gamma(\lambda) = \mathcal{Z}(\lambda)\Gamma$,

$$\mathcal{Z}(\lambda) = \begin{pmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

obtained by rotation of Γ of the angle λ about the x_3-axis. We assume that λ is so small ($|\lambda| \leq \lambda_1$) that $\Gamma(\lambda)$ is contained in the δ_1- neighborhood of $\mathcal{G}$ and can be represented in the form

$$X = z + N(z)\rho(z,\lambda), \quad z \in \mathcal{G}_0. \tag{2.5}$$

We need to compute the derivative $\rho_\lambda(z,\lambda)$. It was done in [2]; here a more elementary representation formula for this function is given. Let $\sigma = (\sigma_1, \sigma_2)$ be local coordinates at the point z. According to formula (3.17) in [2],

$$\frac{\partial \rho(z(\sigma),\lambda)}{\partial \lambda} = \sum_{k=1}^{3} \Big(N_k(\sigma) - \sum_{\beta=1}^{2} \frac{\partial \widetilde{\rho}(\sigma,\lambda)}{\partial \sigma_\beta} J^{\beta k}(\sigma,\rho) \Big) (e_3 \times X)_k$$

where $J^{\beta k}(\sigma,\rho)$ are elements of the matrix inverse to (2.2) with s,r replaced by σ,ρ. Computations of $J^{\beta k}$ shows that

$$\sum_{\beta=1}^{2}\frac{\partial\rho(\sigma,\lambda)}{\partial\sigma_\beta}J^{\beta k}(\sigma,\rho)=\frac{(\rho_{,\sigma_1}X_{,\sigma_2}-\rho_{,\sigma_2}X_{,\sigma_1})\times N(z(\sigma))}{\det\mathcal{J}(\sigma,\rho(\sigma,\lambda))}$$

where

$$\rho_{,\sigma_\alpha}=\frac{\partial\rho}{\partial\sigma_\alpha},\quad X_{,\sigma_j}=\frac{\partial z(\sigma)}{\partial\sigma_j}+\frac{\partial N(\sigma)}{\partial\sigma_j}\rho.$$

By virtue of (2.3),

$$N-\frac{(\rho_{,\sigma_1}X_{,\sigma_2}-\rho_{,\sigma_2}X_{,\sigma_1})\times N(z(\sigma))}{\det\mathcal{J}(\sigma,\rho(\sigma,\lambda))}=\frac{\frac{dX}{d\sigma_1}\times\frac{dX}{d\sigma_2}}{\det\mathcal{J}(\sigma,\rho(\sigma,\lambda))}$$

where $\frac{dX}{d\sigma_j}=X_{,\sigma_j}+N\rho_{\sigma_j}$. In addition, we have

$$\det\mathcal{J}(\sigma,\rho(\sigma,\lambda))=N\cdot(X_{,\sigma_1}\times X_{,\sigma_2})=N\cdot\Big(\frac{dX}{d\sigma_1}\times\frac{dX}{d\sigma_2}\Big)=N\cdot n\Big|\frac{dX}{d\sigma_1}\times\frac{dX}{d\sigma_2}\Big|$$

where $n=n(X)$ is the exterior normal to $\Gamma(\lambda)$ at the point X; hence,

$$\rho_\lambda(z,\lambda)=\frac{n(X)\cdot(e_3\times X)}{N(z)\cdot n(X)}. \tag{2.6}$$

It is possible to obtain a more explicit representation of ρ_λ in terms of ρ that is convenient for further calculations. Let us consider again the surface Γ given by (2.4) and let us define a mapping

$$x=y+N(y)\rho(y)\equiv e_\rho(y):\mathcal{F}\to\Omega,\quad y\in\mathcal{F},$$

where Ω is a domain bounded by Γ, and $N(y)$, $\rho(y)$ are extended from $\mathcal{G}$ into $\mathcal{F}$ in such a way that

$$\frac{\partial N(y)}{\partial N}\Big|_{\mathcal{G}}=0,\quad \frac{\partial\rho(y)}{\partial N}\Big|_{\mathcal{G}}=0. \tag{2.7}$$

We introduce the following notations: $\mathcal{L}=\frac{\partial e_\rho}{\partial y}$ is the Jacobi matrix of the transformation e_ρ with the elements

$$l_{ij}=\delta_{ij}+\frac{\partial}{\partial y_j}N_i(y)\rho(y,t),$$

$L_\rho=\det\mathcal{L}$, l^{ij} are elements of the inverse matrix $\mathcal{L}^{-1}$, $\widehat{L}_{ij}=L_\rho l^{ij}$ are elements of the adjugate matrix $\widehat{\mathcal{L}}$.

The following formulas are useful for subsequent calculations: the normal $n(x)$ to Γ and the normal $N(y)$ to $\mathcal{G}$ are related to each other by

$$n=\frac{\widehat{\mathcal{L}}^TN}{|\widehat{\mathcal{L}}^TN|} \tag{2.8}$$

where the superscript T denotes transposition; $\widehat{\mathcal{L}}^T N$ is the second order polynomial of ρ and of its first derivatives, hence,

$$\widehat{\mathcal{L}}^T N = N + \delta_0 \widehat{\mathcal{L}}^T N + \frac{1}{2}\delta_0^2 \widehat{\mathcal{L}}^T N$$

where $\delta_0^j \widehat{\mathcal{L}}^T N$, $j = 1, 2$, are the first and the second variations of $\widehat{\mathcal{L}}^T N$ with respect to ρ computed at $r = 0$:

$$\delta_0^j \widehat{\mathcal{L}}^T[\rho]N = \frac{d^j}{ds^j}\widehat{\mathcal{L}}^T[r + s\rho]N\Big|_{s=0,r=0}.$$

Calculations carried out in [6] lead to the formula

$$\sum_{i=1}^{3} \widehat{L}_{ij} N_i = N_j \Lambda(y,\rho) - \frac{\partial \rho}{\partial y_j}(1 - \rho\mathcal{H}(y)) + \rho \sum_{m=1}^{3} \frac{\partial \rho}{\partial y_m}\frac{\partial N_m}{\partial y_j}, \quad y \in \mathcal{G},$$

where

$$\Lambda(y,\rho) = 1 - \rho\mathcal{H}(y) + \rho^2\mathcal{K}(y),$$

i.e.

$$\widehat{\mathcal{L}}^T N = N(y)\Lambda(y,\rho(y)) - (1 - \rho\mathcal{H}(y))\nabla_{\mathcal{G}}\rho + \rho(\nabla_{\mathcal{G}} \otimes N)\nabla_{\mathcal{G}}\rho. \tag{2.9}$$

It follows that $N \cdot \widehat{\mathcal{L}}^T N = \Lambda(y,\rho)$.

We apply (2.9) to the surface

$$\Gamma(\lambda) = \{X = z + N(z)\rho(z,\lambda), \quad z \in \mathcal{G}\}$$

and we transform (2.6) as follows:

$$\rho_\lambda(z,\lambda) = \frac{\widehat{\mathcal{L}}^T N \cdot (e_3 \times X(z))}{N(z) \cdot \widehat{\mathcal{L}}^T N} = N(z) \cdot (e_3 \times X(z))$$

$$-\frac{e_3 \times X(z)}{\Lambda(z,\rho(z,\lambda))}\Big((1 - \rho(z,\lambda)\mathcal{H}(z))\nabla_{\mathcal{G}}\rho(z,\lambda) - \rho(z,\lambda)(\nabla_{\mathcal{G}} \otimes N(z))\nabla_{\mathcal{G}}\rho(z,\lambda)\Big).$$

Since

$$N(z) \cdot (e_3 \times X(z)) = N(z) \cdot (e_3 \times z) \equiv h_0(z)$$

and $N(z) \cdot \nabla_{\mathcal{G}}\rho(z,\lambda) = 0$, we have

$$\frac{\partial \rho(z,\lambda)}{\partial \lambda} = h_0(z) + h(z,\rho(z,\lambda)) \cdot \nabla_{\mathcal{G}}\rho(z,\lambda) \tag{2.10}$$

where

$$h(z,\rho) = -\frac{e_3 \times X(z) - N(z)h_0(z)}{\Lambda(z,\rho)}(1 - \rho\mathcal{H}(z)) + \rho(z,\lambda)\frac{((e_3 \times X(z)) \cdot \nabla_{\mathcal{G}})N(z)}{\Lambda(z,\rho)} \tag{2.11}$$

is a smooth function of ρ, if

$$|\rho| \le \delta \ll 1.$$

It is easily verified that $N(z) \cdot h(z) = 0$.

We are looking for the value λ_0 of λ for which the equation

$$f(\lambda) = \int_{\mathcal{G}} \rho(z,\lambda)\rho_\lambda(z,\lambda)dS = 0 \tag{2.12}$$

holds (in [2] another equation $\int_{\mathcal{G}} h_0(z)\rho(z,\lambda)dS = 0$ was considered). The following proposition is an analogue of lemma 3.1 in [2].

Proposition 2.1. *There exist positive constants δ, $\lambda_2 \in (0,\lambda_1]$ and ϵ_1 depending only on $\mathcal{G}$, such that if*

$$|\rho|_{C^1(\mathcal{G})} \le \delta, \tag{2.13}$$

$|\lambda| \le \lambda_2$ *and*

$$\|\rho\|_{L_2(\mathcal{G}} \le \epsilon_1, \quad \epsilon_1 > 0,$$

then

$$f'_\lambda(\lambda) \ge \frac{1}{2}\int_{\mathcal{G}} h_0^2(y)dS, \tag{2.14}$$

and equation (2.12) has a unique solution in the interval $|\lambda| \le \lambda_2$.

Proof. It is clear that $f(\lambda_0) = 0$ is equivalent to

$$f(0) = -\int_0^{\lambda_0} f_\lambda(\lambda)d\lambda.$$

Let us estimate from below the derivative

$$f_\lambda(\lambda) = \int_{\mathcal{G}} (\rho_\lambda^2(z,\lambda) + \rho(z,\lambda)\rho_{\lambda\lambda}(z,\lambda))dS. \tag{2.15}$$

We have

$$\int_{\mathcal{G}} \rho_\lambda^2(z,\lambda)dS = \int_{\mathcal{G}} \Big(h_0^2(z) + (h(z,\rho(z,\lambda) \cdot \nabla_{\mathcal{G}}\rho(z,\lambda))^2\Big)dS$$

$$+2\int_{\mathcal{G}} h_0(z)h(z,\rho(z,\lambda)) \cdot \nabla_{\mathcal{G}}\rho(z,\lambda)dS.$$

Since

$$h(z,\rho) \cdot \nabla_{\mathcal{G}}\rho = \nabla_{\mathcal{G}} \cdot \int_0^\rho h(z,r)dr - \int_0^\rho \nabla_{\mathcal{G}} \cdot h(z,r)dr,$$

integration by parts leads to

$$\int_{\mathcal{G}} h_0 h \cdot \nabla_{\mathcal{G}}\rho dS = -\int_{\mathcal{G}} \nabla_{\mathcal{G}} h_0 \cdot \Big(\int_0^\rho h(z,r)dr\Big)dS - \int_{\mathcal{G}} h_0 \Big(\int_0^\rho \nabla_{\mathcal{G}} \cdot h(z,r)dr\Big)dS,$$

which implies

$$\Big|\int_{\mathcal{G}} h_0 h \cdot \nabla_{\mathcal{G}}\rho dS\Big| \le c\int_{\mathcal{G}} |\rho(z,\lambda)|dS,$$

if δ is sufficiently small.

The second derivative $\rho_{\lambda\lambda}(z,\lambda)$ is given by the formula

$$\rho_{\lambda\lambda}(z,\lambda) = h(z,\rho)\cdot\nabla_{\mathcal{G}}\frac{\partial\rho}{\partial\lambda} + \frac{\partial\rho}{\partial\lambda}h_{\rho}(z,\rho)\cdot\nabla_{\mathcal{G}}\rho, \tag{2.16}$$

hence,

$$\int_{\mathcal{G}}\rho(z,\lambda)\rho_{\lambda\lambda}(z,\lambda)dS = -\int_{\mathcal{G}}\nabla_{\mathcal{G}}\cdot(\rho h(z,\rho))\rho_{\lambda}dS + \int_{\mathcal{G}}\rho\rho_{\lambda}h_{\rho}(z,\rho)\cdot\nabla_{\mathcal{G}}\rho dS$$

and for small δ

$$f_{\lambda}(z,\lambda) \geq \int_{\mathcal{G}} h_0^2(z)dS - c_1\|\rho(\cdot,\lambda)\|_{W_1^1(\mathcal{G})} - c_2\|\rho(\cdot,\lambda)\|^2_{W_2^1(\mathcal{G})}.$$

Since

$$\rho(y) = R(x), \quad \rho(z,\lambda) = R(\mathcal{Z}(\lambda)x),$$

we have

$$|\rho(z,\lambda)-\rho(y)| \leq |(\mathcal{Z}(\lambda)-I)x| \leq c|\lambda|.$$

The derivatives $\rho_{,\sigma_j}$ are related to $\nabla_{\mathcal{G}}\rho$ by

$$\rho_{,\sigma_\alpha} = \nabla_{\mathcal{G}}\rho\cdot\frac{\partial z}{\partial\sigma_\alpha}, \quad \alpha=1,2, \quad 0 = \nabla_{\mathcal{G}}\rho\cdot N(\sigma),$$

hence,

$$\nabla_{\mathcal{G}}\rho = \mathcal{J}^{-1}(\sigma,0)\Big(\rho_{,\sigma_1},\rho_{,\sigma_2},0\Big)^T,$$

and from inequality (3.42) in [2] it follows that

$$|\nabla_{\mathcal{G}}\rho(z,\lambda) - \nabla_{\mathcal{G}}\rho(y)| \leq c|\lambda|$$

which implies

$$f_{\lambda}(\lambda) \geq \int_{\mathcal{G}} h_0^2(z)dS - c_3\delta - c_4|\lambda|.$$

Since $\mathcal{G}$ is not rotationally symmetric, the integral $\int_{\mathcal{G}} h_0^2(z)dS$ is positive, and if

$$c_3\delta \leq \frac{1}{4}\int_{\mathcal{G}} h_0^2(z)dS, \quad c_4\lambda_2 \leq \frac{1}{4}\int_{\mathcal{G}} h_0^2(z)dS,$$

then (2.14) holds for $|\lambda| \leq \lambda_2$. Finally,

$$|f(0)| = \Big|\int_{\mathcal{G}}\rho\rho_{\lambda}|_{\lambda=0}dS\Big| \leq c_5\epsilon_1,$$

so in the case

$$c_5\epsilon_1 \leq \frac{\lambda_2}{2}\int_{\mathcal{G}} h_0^2(z)dS$$

equation (2.12) has a unique solution in the interval $|\lambda| \le \lambda_2$. The proposition is proved.

Equation (2.12) and inequality (2.14) mean that the functional $\int_{\mathcal{G}} \rho^2(z,\lambda)dS$ takes a minimal value for $\lambda = \lambda_0$, and this minimum is unique in the interval $|\lambda| \le \lambda_2$.

Assume finally that there is given a one-parameter family of surfaces Γ_t, $t \in [0, t_0]$ (e.g. Γ_t in problem (1.2)) and each Γ_t is given by equation (2.4) where $\rho = \rho(y,t)$ satisfies (2.13) and is differentiable with respect to t. As above, we consider the surfaces $\Gamma_t(\lambda) = \mathcal{Z}(\lambda)\Gamma_t$ given by equation (2.5) with $\rho = \rho(y,t,\lambda)$, $y \in \mathcal{G} \equiv \mathcal{G}_0$, and we look for the value $\lambda(t)$ of the angle λ such that

$$f(\lambda,t) \equiv \int_{\mathcal{G}} \rho(z,t,\lambda)\rho_\lambda(z,t,\lambda)dS = 0. \tag{2.17}$$

If $\rho(y,t)$ satisfies the hypotheses of proposition 2.1 for all $t \in [0,t_0]$, then such a function exists and satisfies the inequality

$$|\lambda(t)| \le \lambda_2.$$

Moreover, differentiation of (2.17) with respect to t leads to

$$\lambda_t(t) = -\frac{f_t(\lambda,t)}{f_\lambda(\lambda,t)}\Big|_{\lambda=\lambda(t)} = -\frac{\int_{\mathcal{G}}(\rho_t\rho_\lambda + \rho\rho_{\lambda t})dS}{\int_{\mathcal{G}}(\rho_\lambda^2 + \rho\rho_{\lambda\lambda})dS}\Big|_{\lambda=\lambda(t)}. \tag{2.18}$$

It is important to emphasize that the derivative ρ_t and $\rho_{\lambda t}$ in (2.18) should be computed with λ fixed. By (2.10),

$$\rho_{\lambda t}(z,t,\lambda) = h(z,\rho(z,t,\lambda))\cdot\nabla_{\mathcal{G}}\rho_t(z,t,\lambda) + \rho_t(z,t,\lambda)h_\rho(z,\rho(z,t,\lambda))\cdot\nabla_{\mathcal{G}}\rho(z,t,\lambda). \tag{2.19}$$

The second derivative $\lambda_{tt}(t)$ is given by

$$\begin{aligned}\lambda_{tt}(t) &= -\Big(\frac{\partial}{\partial t}\frac{f_t(\lambda,t)}{f_\lambda(\lambda,t)}\Big)_{\lambda=\lambda(t)} - \lambda_t(t)\Big(\frac{\partial}{\partial\lambda}\frac{f_t(\lambda,t)}{f_\lambda(\lambda,t)}\Big)_{\lambda=\lambda(t)} \\ &= -\Big(\frac{f_{tt}}{f_\lambda} - \frac{f_t f_{t\lambda}}{f_\lambda^2}\Big)_{\lambda=\lambda(t)} - \Big(\frac{f_{t\lambda}}{f_\lambda} - \frac{f_t f_{\lambda\lambda}}{f_\lambda^2}\Big)_{\lambda=\lambda(t)}\lambda_t(t)\end{aligned} \tag{2.20}$$

where

$$\begin{aligned}f_{tt}(\lambda,t) &= \int_{\mathcal{G}}(\rho_{tt}\rho_\lambda + \rho\rho_{tt\lambda} + 2\rho_t\rho_{t\lambda})dS, \\ f_{t\lambda}(\lambda,t) &= \int_{\mathcal{G}}(\rho_{\lambda\lambda}\rho_t + \rho\rho_{t\lambda\lambda} + 2\rho_\lambda\rho_{t\lambda})dS, \\ f_{\lambda\lambda}(\lambda,t) &= \int_{\mathcal{G}}(\rho\rho_{\lambda\lambda\lambda} + 3\rho_\lambda\rho_{\lambda\lambda})dS.\end{aligned} \tag{2.21}$$

3. Transformation of problem (1.2).

Let us consider free boundary problem (1.2) under the following assumptions concerning the initial data:

1. Γ_0 is given by (1.3) with $\rho = \rho_0 \in C^{3+\alpha}(\mathcal{G}_0)$ satisfying (2.13) and (2.12), i.e.

$$\int_{\mathcal{G}_0} \rho_0(y)(h_0(y) + h(y, \rho_0(y)) \cdot \nabla_{\mathcal{G}} \rho_0(y) dS = 0; \tag{3.1}$$

2. $v_0 \in C^{2+\alpha}(\Omega_0)$ satisfies (1.4) and the compatibility conditions

$$\nabla \cdot v_0(y) = 0, \qquad S(v_0)n_0 - n_0(n_0 \cdot S(v_0)n_0) = 0, \quad y \in \Omega_0$$

where n_0 is the exterior normal to Γ_0;

3. The smallness condition

$$\|\vec{v}_0\|_{L_2(\Omega_0)} + \|\rho_0\|_{L_2(\mathcal{G}_0)} \leq \epsilon_1 \tag{3.2}$$

is satisfied with the number ϵ_1 chosen in proposition 2.1 (below some other restrictions on ϵ_1 are imposed). Then, according to theorem 4.2 in [2], problem (1.7) has a unique solution $v(\cdot, t) \in C^{2+\alpha}(\Omega_t)$, $\nabla p(\cdot, t) \in C^{\alpha}(\Omega_t)$ with $v_t(\cdot, t) \in C^{\alpha}(\Omega_t)$, defined for $t \in [0, t_0]$, the surface Γ_t is representable in the form (2.4) with $\rho = \rho(\cdot, t) \in C^{3+\alpha}(\mathcal{G}_0)$, having the derivatives $\rho_t(\cdot, t) \in C^{2+\alpha}(\mathcal{G}_0)$, $\rho_{tt}(\cdot, t) \in C^{\alpha}(\mathcal{G}_0)$, and the solution satisfies the inequalities

$$\sup_{t<t_0} |v_t(\cdot, t)|_{C^{\alpha}(\Omega_t)} + \sup_{t<t_0} |v(\cdot, t)|_{C^{2+\alpha}(\Omega_t)} + \sup_{t<t_0} |\nabla p(\cdot, t)|_{C^{1+\alpha}(\Omega_t)}$$

$$+ \sup_{t<t_0} |\rho(\cdot, t)|_{C^{3+\alpha}(\mathcal{G}_0)} + \sup_{t<t_0} |\rho(\cdot, t)|_{C^{2+\alpha}(\mathcal{G}_0)} + \sup_{t<t_0} |\rho_{tt}(\cdot, t)|_{C^{\alpha}(\mathcal{G}_0)} \leq$$

$$\leq c\Big(|v_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{3+\alpha}(\mathcal{G}_0)}\Big).$$

Moreover, there exists a function $\vartheta \in C^2([0, t_0])$ such that Γ_t is given by (1.11) with the function $\widehat{\rho}$ possessing the same regularity properties as ρ, satisfying the inequality

$$\sup_{t<t_0} |\widehat{\rho}(\cdot, t)|_{C^{3+\alpha}(\mathcal{G}_{\vartheta(t)})} + \sup_{t<t_0} |\widehat{\rho}(\cdot, t)|_{C^{2+\alpha}(\mathcal{G}_{\vartheta(t)})} + \sup_{t<t_0} |\widehat{\rho}_{tt}(\cdot, t)|_{C^{\alpha}(\mathcal{G}_{\vartheta(t)})} \leq$$

$$\leq c\Big(|v_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{3+\alpha}(\mathcal{G}_0)}\Big)$$

and the condition (2.12), i.e.

$$\int_{\mathcal{G}_{\vartheta(t)}} \widehat{\rho}(\xi, t)(h_0(\xi) + h(\xi, \widehat{\rho}(\xi, t)) \cdot \nabla_{\mathcal{G}_{\vartheta(t)}} \widehat{\rho}(\xi, t) dS = 0. \tag{3.3}$$

The existence of $\vartheta(t)$ follows from proposition 2.1; it is related to $\lambda(t)$ by

$$\vartheta(t) = -\lambda(t)$$

and $\widehat{\rho}$ is defined by

$$\widehat{\rho}(\xi, t) = \rho(\mathcal{Z}(\lambda(t))\xi, t, \lambda(t)) \equiv \widetilde{\rho}(\mathcal{Z}(\lambda(t))\xi, t). \tag{3.4}$$

The regularity of $\widehat{\rho}(\xi,t)$ as a function of ξ follows from the boundary condition

$$n \cdot T(v,p)n = \sigma H + \frac{\omega^2}{2}|x'|^2 + \kappa U + p_0$$

that can be written in the form

$$\sigma(H(x) - \widehat{\mathcal{H}}(\xi)) + \frac{\omega_0^2}{2}(|x^{'2}| - |\xi^{'2}|) + \kappa(U(x,t) - \widehat{\mathcal{U}}(\xi)) = \vec{n} \cdot T(\vec{w}.s)\vec{n}(x),$$

where $\xi \in \mathcal{G}_{\vartheta(t)}$, $x = \xi + \widehat{N}(\xi)\widehat{\rho}(\xi,t) \in \Gamma_t$, $\widehat{\mathcal{H}}(\xi)$ is the doubled mean curvature of $\mathcal{G}_{\theta(t)}$ at the point ξ, and $\widehat{\mathcal{U}}(\xi) = \int_{\mathcal{F}_{\theta(t)}} |\xi - \eta|^{-1} d\eta$ (see [2], (4.18), and [7], proposition 3.1). The derivatives of $\lambda(t)$ are given by (2.18), (2.20). By (3.4), $\rho_t(z,t,\lambda(t))$ in these formulas coincides with the derivative $\widehat{\rho}_t(\xi,t)$ computed for ξ fixed. By virtue of the kinematic boundary condition $V_n = v \cdot n$ on Γ_t, we have

$$\widehat{\rho}_t(\xi,t) = \frac{v(x,t) \cdot n(x)}{\widehat{N}(\xi) \cdot n(x)} \tag{3.5}$$

where $\xi \in \mathcal{G}_{\vartheta(t)}$, $x = \xi + \widehat{N}(\xi)\widehat{\rho}(\xi,t) \in \Gamma_t$.

Equation (3.3) is equivalent to (2.12), and due to (2.14) it means that the integral

$$I(\vartheta(t)) = \int_{\mathcal{G}_{\vartheta(t)}} \widehat{\rho}^2(\xi,t)dS$$

takes a minimal value in comparison with all the integrals

$$I(\theta) = \int_{\mathcal{G}_\theta} \rho_\theta^2(y)dS$$

such that

$$\Gamma_t = \{x = y + N_\theta(y)\rho_\theta(y), \quad y \in \mathcal{G}_\theta\},$$

at least if $|\theta - \vartheta(t)| \le \lambda_2$.

Let us choose $\mathcal{G}_0$ in such a way that the integral $\int_{\mathcal{G}_0} \rho_0^2(y)dS$ is minimal among all the integrals $\int_{\mathcal{G}_0} \rho^2(z)dS$ such that

$$\Gamma_0 = \{x = z + N_\theta(z)\rho(z), \quad z \in \mathcal{G}_\theta\}.$$

Then, if we take ϵ_1 in (3.2) sufficiently small, all the integrals $I(\vartheta(t))$, $t \in [0,t_0]$, will have the same minimal property with respect to the surfaces Γ_t. Indeed, in the opposite case there would exist θ with $|\theta - \vartheta(t)| \ge \lambda_2$ such that $I(\theta(t)) \ge I(\theta)$. Of course, this is always true for $\theta = \vartheta(t) + 2\pi m$ with m integer or, if $\mathcal{F}$ is periodic with respect to the angle of rotation about the x_3-axis with a minimal period $2\pi/k$ (k is also an integer), for $\theta = \vartheta(t) + 2\pi m/k$. But since $\mathcal{G}_\theta$ are not rotationally symmetric, $\mathcal{G}_{\theta_1}$ is different from $\mathcal{G}_{\theta_2}$, if $|\theta_1 - \theta_2| > 2m\pi/k$, and it is possible to choose ϵ_1 so small that inequalities $\epsilon_1 \le I(\theta_1)$ and $\epsilon_1 \le I(\theta_2)$ can not hold simultaneously, if $|\theta_1 - \theta_2 - 2\pi m/k| \ge \lambda_2$. In what follows we assume that this condition is satisfied, and then

$$I(\vartheta(t)) \le \epsilon_1 \tag{3.6}$$

and

$$I(\vartheta(t)) \leq I(\theta) \tag{3.7}$$

for all the possible θ.

Now, we assume that the solution of problem (1.2) with the properties indicated above exists in an infinite time interval $t > 0$, and that Γ_t stays always near the set $\cup_\theta \mathcal{G}_\theta$, in particular,

$$\|v(\cdot,t)\|^2_{L_2(\Omega_t)} + I(\theta) \leq \epsilon_1$$

for a certain θ with ϵ_1 sufficiently small. Then the function $\lambda(t)$ (and also $\vartheta(t) = -\lambda(t)$) that has been defined for $t \in [0, t_0]$ can be extended into the whole interval $t \in [0, \infty)$, and (2.10), (2.17)-(2.21), (3.6), (3.7) hold in this interval. Indeed, we have $\|v(\cdot,t_0)\|^2_{L_2(\Omega_{t_0})} + I(\vartheta(t_0)) \leq \epsilon_1$ and we can estimate higher order norms of $(v, p, \widehat{\rho})$ at $t = t_0$ by theorem 4.4 in [2]. Then we can construct $\lambda(t)$ and $\widehat{\rho}(\xi, t)$ satisfying the inequality $|\widehat{\rho}(\cdot,t)|_{C^1(\mathcal{G}_{\vartheta(t)})} \leq \delta$ for $t \in [t_0, 2t_0]$ etc.

Let us transform problem (1.2). At first we make the change of variables

$$z = \mathcal{Z}(\lambda(t))x$$

that maps Ω_t onto $\widetilde{\Omega}_t = \mathcal{Z}(\lambda(t))\Omega_t$ and Γ_t onto $\widetilde{\Gamma}_t = \partial\widetilde{\Omega}_t$, and we introduce the functions

$$w(z,t) = \mathcal{Z}(\lambda(t))v(\mathcal{Z}^{-1}(\lambda(t))z,t), \quad s(z,t) = p(\mathcal{Z}^{-1}(\lambda(t))z,t).$$

An elementary calculation shows that w and s satisfy the relations

$$\begin{gathered}
w_t + (w\cdot\nabla)w + 2\omega(e_3\times w) - \lambda_t(t)(e_3\times w) + \lambda_t(t)(\eta_3(z)\cdot\nabla)w - \nu\nabla^2 w + \nabla s = 0,\\
\nabla\cdot w = 0, \qquad z\in\widetilde{\Omega}_t, \qquad t>0,\\
T(w,s)\widetilde{n} = (\sigma\widetilde{H}(z) + \frac{\omega^2}{2}|z'|^2 + p_0 + \kappa\widetilde{U}(z,t))\widetilde{n}, \\
V_n = w\cdot\widetilde{n} + \lambda_t(t)\eta_3(z)\cdot\widetilde{n}(z), \qquad z\in\widetilde{\Gamma}_t\\
w(z,0) = v_0(z), \qquad z\in\Omega_0.
\end{gathered} \tag{3.8}$$

Here $\widetilde{n}$ is the exterior normal to $\widetilde{\Gamma}_t$, $\widetilde{H}(z)$ is the doubled mean curvature of $\widetilde{\Gamma}_t$ and $\widetilde{U}(y,t) = \int_{\widetilde{\Omega}_t} |y-z|^{-1}dy$. The surface $\widetilde{\Gamma}_t$ is given by

$$z = y + N(y)\widetilde{\rho}(y,t), \quad y\in\mathcal{G}\equiv\mathcal{G}_0$$

where $\widetilde{\rho}(y,t) = \rho(y,t,\lambda(t))$. The orthogonality conditions (1.8) are invariant:

$$\int_{\widetilde{\Omega}_t} w(z,t)dz = 0,$$

$$\int_{\widetilde{\Omega}_t} w(z,t)\cdot\eta_i(z)dz + \omega\int_{\widetilde{\Omega}_t}\eta_3(z)\cdot\eta_i(z)dz = \omega\int_{\mathcal{F}}\eta_3(z)\eta_i(z)dz, \qquad i=1,2,3. \tag{3.9}$$

Conditions (1.6) are equivalent to

$$\int_{\mathcal{G}} \varphi(y,\widetilde{\rho})dS = 0, \qquad \int_{\mathcal{G}} \varphi(y,\widetilde{\rho})y_i dS = -\int_{\mathcal{G}} N_i(y)\Big(\frac{\widetilde{\rho}^2}{2} - \frac{\widetilde{\rho}^3}{3}\mathcal{H}(y) + \frac{\widetilde{\rho}^4}{4}\mathcal{K}(y)\Big)dS, \tag{3.10}$$

where

$$\varphi(y,\rho) = \rho - \frac{\rho^2(y)}{2}\mathcal{H}(y) + \frac{\rho^3(y)}{3}\mathcal{K}(y).$$

As in [5], we transform free boundary problem (3.8) into a nonlinear problem in a given domain $\mathcal{F} \equiv \mathcal{F}_0$. We extend $N(y)$ and $\widetilde{\rho}(y,t)$ from $\mathcal{G}$ into $\mathcal{F}$ in such a way that N remains smooth (for our purposes it is sufficient that $N \in C^{3+\alpha}(\mathcal{F})$) and $\widetilde{\rho}$ satisfies the inequalities

$$|\widetilde{\rho}(\cdot,t)|_{C^{3+\alpha}(\mathcal{F})} \leq c|\widetilde{\rho}(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})},$$

$$|\widetilde{\rho}_t(\cdot,t)|_{C^{2+\alpha}(\mathcal{F})} \leq c|\widetilde{\rho}_t(\cdot,t)|_{C^{2+\alpha}(\mathcal{G})},$$

$$|\widetilde{\rho}(\cdot,t)|_{C^1(\mathcal{F})} \leq \delta \ll 1.$$

Finally, both N and $\widetilde{\rho}$ should satisfy (2.7). Now, we map $\mathcal{F}$ onto $\widetilde{\Omega}_t$ by the transformation

$$z = y + N(y)\widetilde{\rho}(y,t) \equiv \widetilde{e}_\rho(y), \qquad y \in \mathcal{F} \tag{3.11}$$

that is invertible if δ is small enough, and we pass in (3.8) to the variables $y \in \mathcal{F}$. As above in Sect.2, we introduce the following notations: $\mathcal{L} = \frac{\partial \widetilde{e}_\rho(y)}{\partial y}$ is the Jacobi matrix of the transformation $e(y)$ with the elements

$$l_{ij} = \delta_{ij} + \frac{\partial}{\partial y_j}N_i(y)\widetilde{\rho}(y,t),$$

$\widetilde{L}_\rho = \det\mathcal{L}$, l^{ij} are elements of the inverse matrix $\mathcal{L}^{-1}$, $\widehat{L}_{ij} = \widetilde{L}_\rho l^{ij}$ are elements of the adjugate matrix $\widehat{\mathcal{L}}$. The change of variables (3.11) transforms ∇_z into $\widetilde{\nabla} = \mathcal{L}^{-T}\nabla_y$ (the superscript T means transposition, $\mathcal{L}^{-T} = (\mathcal{L}^{-1})^T$). Repeating the arguments in [5] where mapping (3.11) was applied to problem (1.2) we show that (3.8) is transformed into

$$\begin{gathered}
u_t + 2\omega(e_3 \times u) - \nu\nabla^2 u + \nabla q = \widetilde{f}(w,s,\widetilde{\rho}),\\
\nabla \cdot u = 0, \qquad y \in \mathcal{F},\\
T(u,q)N(y) + NB_0\widetilde{\rho} = \nu b(w,\widetilde{\rho}) + Nd(w,\widetilde{\rho}),\\
\widetilde{\rho}_t(y,t) = u(y,t)\cdot N(y) - \frac{h_0(y)}{\int_{\mathcal{G}} h_0^2(z)dS}\int_{\mathcal{G}} u(z,t)\cdot N(z)h_0(z)dS + g(w,\widetilde{\rho}),\\
\widetilde{\rho}(y,0) = \rho_0(y), \qquad y \in \mathcal{G},\\
u(y,0) = w_0(y), \qquad y \in \mathcal{F},
\end{gathered} \tag{3.12}$$

where

$$u(y,t) = \widehat{\mathcal{L}}w(\widetilde{e}_\rho(y),t), \quad q(y,t) = s(\widetilde{e}_\rho(y),t),$$

$$B_0\widetilde{\rho} = -\sigma\Delta_{\mathcal{G}}\widetilde{\rho} - \beta(x)\widetilde{\rho}(x) - \kappa\int_{\mathcal{F}}\frac{\widetilde{\rho}(z)dz}{|x-z|},$$

$\Delta_{\mathcal{G}}$ is the Laplace-Beltrami operator on $\mathcal{G}$, $\beta(y)$ is defined in (1.8) and $f,\ b,\ d,\ g$ are the following nonlinear functions of $u,\ q,\ \widetilde{\rho}$ and of their derivatives:

$$\widetilde{f} = (u_t - \widetilde{L}_\rho^{-1}\mathcal{L}u_t) - (\widetilde{L}_\rho^{-1}\mathcal{L})_t u + \rho_t(\mathcal{L}^{-1}N\cdot\nabla)(\widetilde{L}_\rho^{-1}\mathcal{L}u) - \widetilde{L}_\rho^{-1}(w\cdot\nabla)(\widetilde{L}_\rho^{-1}\mathcal{L}u)$$

$$+2\omega(e_3\times(u-\widetilde{L}_\rho^{-1}\mathcal{L}u)) + \nu\Big(\widetilde{\nabla}\cdot\widetilde{\nabla}(\widetilde{L}_\rho^{-1}\mathcal{L}u) - \nabla^2 u\Big) + (\nabla-\widetilde{\nabla})q$$

$$+\lambda_t(t)(e_3\times L_\rho^{-1}\mathcal{L}u) - \lambda_t(t)(\eta_3(\widetilde{e}_\rho(y))\cdot\widetilde{\nabla})L_\rho^{-1}\mathcal{L}u,$$

$$b(u,\widetilde{\rho}) = \Pi_0(\Pi_0 S(u)N - \Pi\widetilde{S}(\widetilde{L}_\rho^{-1}\mathcal{L}u)\widetilde{n}),$$

$$d(u,\widetilde{\rho}) = \nu d_1(u,\widetilde{\rho}) + \sigma d_2(\widetilde{\rho}) + \kappa d_3(\widetilde{\rho}),$$

$$d_1(u,\widetilde{\rho}) = \widetilde{n}\cdot\widetilde{S}(\widetilde{L}_\rho^{-1}\mathcal{L}u)\widetilde{n} - N\cdot S(u)N,$$

$$d_2(\widetilde{\rho}) = N\cdot(\Delta_\Gamma - \Delta_{\mathcal{G}} - \delta_0\Delta_\Gamma)y + (\widetilde{n}-N)\cdot\Delta_\Gamma(N\widetilde{\rho})$$

$$+N\cdot(\Delta_\Gamma-\Delta_{\mathcal{G}})(N\widetilde{\rho}) + (\widetilde{n}-N)\cdot(\Delta_\Gamma-\Delta_{\mathcal{G}})y + (\widetilde{n}\cdot N-1)\mathcal{H} + \frac{\omega^2}{2\sigma}(N_1^2+N_2^2)\widetilde{\rho}^2,$$

$$d_3(\widetilde{\rho}) = \int_0^1(1-\mu)d\mu\int_{\mathcal{F}}\frac{d^2}{d\mu^2}\frac{\widetilde{L}_{\mu\rho}(z)}{|\widetilde{e}_{\mu\rho}(y,t)-\widetilde{e}_{\mu\rho}(z,t)|}dz,$$

$$g(u,\widetilde{\rho}) = (\widetilde{\rho}\mathcal{H}(y) - \widetilde{\rho}^2\mathcal{K}(y))w\cdot N + g_1(u,\widetilde{\rho}),$$

$$g_1(u,\widetilde{\rho}) = \lambda_t(t)(\eta_3(\widetilde{e}_\rho)\cdot\widetilde{n} - \eta_3(y)\cdot N(y)) + h_0(y)\Big(\lambda_t(t) + \frac{\int_{\mathcal{G}}u(z,t)\cdot N(z)h_0(z)dS}{\int_{\mathcal{G}}h_0^2(z)dS}\Big). \tag{3.13}$$

Here $\widetilde{S}(w) = \widetilde{\nabla}w + (\widetilde{\nabla}w)^T$, $\widetilde{n} = \widetilde{n}(\widetilde{e}_\rho)$ and Δ_Γ is the Laplace-Beltrami operator on $\widetilde{\Gamma}_t$ whose coefficients depend on $\widetilde{\rho}$. The transformed kinematic boundary condition can be also written in the form

$$\frac{\partial}{\partial t}\varphi(y,\widetilde{\rho}) = u(y,t)\cdot N(y) - \frac{h_0(y)}{\int_{\mathcal{G}}h_0^2(z)dS}\int_{\mathcal{G}}u(z,t)\cdot N(z)h_0(z)dS + g_1(u,\widetilde{\rho}).$$

We recall that $\lambda_t(t)$ is given by (2.18) where

$$\rho_t(z,t,\lambda(t)) = \frac{u(x,y)\cdot N(z)}{\Lambda(z,\widetilde{\rho})},\quad x = z + N(z)\widetilde{\rho},$$

by virtue of (3.5) and (2.8); hence, the expression g_1 is indeed nonlinear (at least quadratic) function of $u,\ q,\ \widetilde{\rho}$ and of their derivatives. The orthogonality conditions (3.9) take the form

$$\int_{\mathcal{F}}udy = \int_{\mathcal{F}}(I-\mathcal{L})udy,$$

$$\int_{\mathcal{F}}u\cdot\eta_i dy + \omega\int_{\mathcal{G}}\widetilde{\rho}\eta_3(y)\cdot\eta_i(y)dS = \int_{\mathcal{F}}u\cdot\eta_i dy - \int_{\mathcal{F}}\mathcal{L}u(y,t)\cdot\eta_i(\widetilde{e}_\rho(y)dy$$

$$-\omega\int_0^1(1-\mu)d\mu\int_{\mathcal{G}}\frac{d}{d\mu}\eta_3(\widetilde{e}_{\mu\rho})\cdot\widetilde{\eta}_i(e_{\mu\rho})\widetilde{\rho}\Lambda(y;\mu\widetilde{\rho})dS. \tag{3.14}$$

Finally, we have the following analogue of proposition 2.2 in [5]:

Proposition 3.1 *Given the functions* $l_i(t),\ m_i(t),\ i=1,2,3,$ *and a tangential vector field* $g(x,t), x\in\mathcal{G},\ t\in[0,T]$, *there exist a function* $r(x,t), x\in\mathcal{G}$ *and a divergence free vector field* $w(x,t),\ x\in\mathcal{F}$, *such that*

$$r_t = w\cdot N - \frac{h_0(x)}{\int_{\mathcal{G}} h_0^2(z)dS}\int_{\mathcal{G}} h_0(z)w\cdot NdS,$$

$$\Pi_0 S(w)N = g(x,t), \qquad x\in\mathcal{G},$$

$$\int_{\mathcal{G}} r(x,t)dS = 0, \qquad \int_{\mathcal{G}} r(x,t)xdS = l(t) = (l_1(t),l_2(t),l_3(t)),$$

$$\int_{\mathcal{F}} w(x,t)dx = l'(t), \qquad \int_{\mathcal{F}} w(x,t)\cdot\eta_i(x)dx = m_i(t), \qquad i=1,2,3,$$

and

$$|r(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})}\le c|l(t)|,$$

$$|w(\cdot,t)|_{C^{2+\alpha}(\mathcal{F})}\le c\Big(|l(t)|+|l'(t)|+|m(t)|+|h|_{C^{1+\alpha}(\mathcal{G})}\Big).$$

The proof is exactly the same as in [5] with the exception of one point: the function r should be taken in the form

$$r(y,t) = |\mathcal{F}|^{-1}l(t)\cdot(N(y) - \frac{h_0(y)}{\int_{\mathcal{G}} h_0^2(z)dS}\int_{\mathcal{G}} h_0(z)N(z)dS.$$

Since $\int_{\mathcal{G}} r\eta_3\cdot\eta_i dS = 0$, condition $\int_{\mathcal{F}} w(x,t)\cdot\eta_i(x)dx = m_i(t)$ implies

$$\int_{\mathcal{F}} w(x,t)\cdot\eta_i(x)dx + \omega\int_{\mathcal{G}} r\eta_3\cdot\eta_i dS = m_i(t).$$

4. Linear problem and instability.

Omitting in (3.12) all the nonlinear terms, we arrive at the linear problem

$$\begin{gathered} v_t + 2\omega(e_3\times v) - \nu\nabla^2 v + \nabla p = 0, \quad \nabla\cdot v = 0, \qquad x\in\mathcal{F}, \\ T(v,p)N + NB_0\rho = 0, \\ \rho_t = v\cdot N - \frac{h_0(x)}{\int_{\mathcal{G}} h_0^2(z)dS}\int_{\mathcal{G}} h_0(z)v(z)\cdot N(z)dS, \quad x\in\mathcal{G} \\ \rho(x,0)=\rho_0(x), \qquad x\in\mathcal{G}, \quad v(x,0)=v_0(x), \qquad x\in\mathcal{F} \end{gathered} \tag{4.1}$$

where unknown are v, p, ρ. It differs from problem (2.23) in [5] by an extra term in the equation for ρ_t that vanishes for axially symmetric $\mathcal{F}$. Linearization of conditions (3.10), (3.14) leads to

$$\int_{\mathcal{G}} \rho dS = 0, \quad \int_{\mathcal{G}} \rho x_i dS = 0, \quad i = 1, 2, 3, \tag{4.2}$$

$$\int_{\mathcal{F}} v dS = 0, \qquad \int_{\mathcal{F}} v \cdot \eta_i dS + \omega \int_{\mathcal{G}} \rho \eta_3 \cdot \eta_i dS = 0, \quad i = 1, 2, 3. \tag{4.3}$$

It is easily verified that if these conditions are satisfied for $t = 0$, they hold also for arbitrary $t > 0$. Along with (4.1), we consider the parameter-dependent problem

$$\begin{gathered} sv + 2\omega(e_3 \times v) - \nu \nabla^2 v + \nabla p = 0, \quad \nabla \cdot v = 0, \qquad x \in \mathcal{F}, \\ T(v, p)N + NB_0\rho = 0, \\ s\rho = v \cdot N - \frac{h_0}{\int_{\mathcal{G}} h_0^2(z) dS} \int_{\mathcal{G}} h_0(z) v(z) \cdot N(z) dS \quad x \in \mathcal{G} \end{gathered} \tag{4.4}$$

that is almost identical to the problem (1.19) in [4]. It is equivalent to the spectral problem for the operator A that acts in the space of functions $\phi = (v, \rho)$ and is defined by

$$A\phi = \Big(A_{11} v + A_{12}\rho, \quad A_{21} v\Big)^T,$$

$$A_{11} v = -2\omega P_J(e_3 \times v) + \nu \nabla^2 v - \nabla s_1, \qquad A_{12}\rho = -\nabla s_2,$$

$$A_{21} v = \Big(v \cdot N - \frac{h_0}{\int_{\mathcal{G}} h_0^2 dS} \int_{\mathcal{G}} h_0 v \cdot N dS\Big)_{\mathcal{G}}.$$

By P_J we mean the orthogonal in $L_2(\mathcal{F})$ projector onto the subspace $J(\mathcal{F}) \subset L_2(\mathcal{F})$ of divergence free vector fields, and s_i are harmonic functions in $\mathcal{F}$ satisfying the conditions

$$s_1(x) = \nu N(x) \cdot S(v) N(x), \qquad s_2 = B_0 \rho(x), \qquad x \in \mathcal{G}.$$

The pressure p is excluded. The domain of A is characterized by the conditions

$$\nabla \cdot v = 0, \qquad \Pi_0 S(v) N(x)|_{\mathcal{G}} = 0$$

and the orthogonality conditions (4.2), (4.3). In fact, it can be shown that they are satisfied for arbitrary solution of equation $A\phi = s\phi$, if $s \neq 0$, $s \neq \pm i\omega$ (see [4], proposition 3.2). Moreover, it can be shown that the spectrum of A consisting of a countable number of eigenvalues of a finite algebraic multiplicity has the following properties:

i). Operator A has no eigenvalues on the imaginary axis, except the point $s = 0$. The corresponding eigenfunctions have the form $\phi_0 = (0, \rho_0)$ with ρ_0 satisfying the equation $\widehat{B}\rho_0 = 0$ where

$$\widehat{B}\rho = B\rho - \frac{1}{|\mathcal{G}|} \int_{\mathcal{G}} B\rho dS,$$

$$B\rho = B_0\rho + \frac{\omega^2|x'|^2}{\int_{\mathcal{G}}|y'|^2 dS}\int_{\mathcal{G}}\rho|y'|^2 dS$$

(in particular, $(0, h_0)$ is an eigenfunction). There are no associated functions corresponding to the eigenvalue $s = 0$, and the dimension of the corresponding root space equals dimKer$\widehat{B}$ (considered on the set of functions satisfying (4.2)).

ii). If the form $\delta_0^2 R[\rho]$ (that coincides with $\int_{\mathcal{G}}\rho\widehat{B}\rho dS$) can take negative values for some ρ satisfying (4.2) and the equilibrium figure $\mathcal{F}$ possess the property

$$\min_{\theta\in[0,2\pi)}\int_{\mathcal{F}}((x_1\cos\theta + x_2\sin\theta)^2 - x_3^2)dS > 0,$$

then the operator A has a finite number of eigenvalues with positive real parts.

Due to the properties

$$B_0 h_0 = \widehat{B}h_0 = 0,$$

$$\int_{\mathcal{G}} h_0 dS = \int_{\mathcal{G}} h_0 x_i dS = 0, \quad \int_{\mathcal{G}} h_0\eta_3\cdot\eta_i dS = 0, \quad i = 1, 2, 3,$$

of the function $h_0 = (e_3 \times x)\cdot N(x)$, the verification of i) and ii) is exactly the same as in [3.4]. It should be observed that all the proofs in [3,4] are done under the hypothesis $\int_{\mathcal{F}} x_1 x_2 dS = 0$ concerning $\mathcal{F}$ but this does not restrict the generality of the results. The proof of ii) in [3] is based on the ideas presented in [8], Ch. 9.

The result of the paper [2] reduces to the following: if h_0 is the only element of Ker$\widehat{B}$, the the equilibrium figure $\mathcal{F}$ is stable.

Let us turn to the proof of instability of the zero solution of problem (3.12), (3.13).

Theorem 4.1. *Assume that $\delta_0^2 R[\rho]$ can take negative values for some ρ satisfying (4.2) and that condition (4.4) is satisfied. Then there exists $\epsilon > 0$ and the non-zero initial data $u_0 \in C^{2+\alpha}(\mathcal{F})$, $\rho_0 \in C^{3+\alpha}(\mathcal{G})$ satisfying the necessary orthogonality and compatibility conditions and having an arbitrarily small norm*

$$|u_0|_{C^{2+\alpha}(\mathcal{F})} + |\rho_0|_{C^{3+\alpha}(\mathcal{G})}$$

such that the solution of (3.12) has the norm

$$|u(\cdot,t)|_{C^{2+\alpha}(\mathcal{F})} + |\widetilde{\rho}(\cdot,t)|_{C^{3+\alpha}(\mathcal{G})} \geq \epsilon \tag{4.5}$$

for certain arbitrarily large $t > 0$.

For problem (1.2), (4.5) implies

$$|v(\cdot,t)|_{C^{2+\alpha}(\Omega_t)} + |\widehat{\rho}(\cdot,t)|_{C^{3+\alpha}(\mathcal{G}_{\vartheta(t)})} \geq \epsilon$$

and, by theorem 4.4 in [2],

$$\sup_{t-\tau<t'<t}\Big(\|v(\cdot,t')\|_{L_2(\mathcal{F})} + \|\widehat{\rho}(\cdot,t')\|_{L_2(\mathcal{G}_{\vartheta(t')})}\Big) \geq c\epsilon$$

for a certain $\tau > 0$ independent of t; by (3.7), this means the lack of stability.

The proof of theorem 4.1 that is almost identical with the proof of proposition 3.1 in [5] is based on the representation formula for the solution of (3.12) that we are going to describe. We set $r_0 = \varphi(\rho_0)$,

$$l = -\int_{\mathcal{G}} N(y)\Big(\frac{\rho_0^2}{2} - \frac{\rho_0^3}{3}\mathcal{H}(y) + \frac{\rho_0^4}{4}\mathcal{K}(y)\Big)dS,$$

$$m_i = \int_{\mathcal{F}} u_0 \cdot \eta_i dy - \int_{\mathcal{F}} \mathcal{L}(y;\rho_0)u(y,0)\cdot \eta_i(e_{\rho_0}(y)dy$$

$$+\int_0^1 (1-\mu)d\mu \int_{\mathcal{G}} \frac{d}{d\mu}\eta_3(e_{\mu\rho_0})\cdot \eta_i(e_{\mu\rho_0})\rho\Lambda(y;\mu\rho_0)dS,$$

$$g(y) = b(u_0,\rho_0) = \Pi_0(\Pi_0 S(u_0)N - \Pi\widetilde{S}(L_{\rho_0}^{-1}\mathcal{L}u_0)n)$$

and we compute the functions u_0'', r_0'' corresponding to these $l, m, g(y)$, according to proposition 3.1. Then $u_0'(y) = u_0 - u_0''$, $r_0'(y) = r_0 - r_0''$ satisfy the conditions

$$\Pi_0 S(u_0')N(x) = 0, \qquad x \in \mathcal{G},$$

$$\int_{\mathcal{G}} r_0' dS = 0, \qquad \int_{\mathcal{G}} r_0' y_i dS = 0,$$

$$\int_{\mathcal{F}} u_0' dy = 0, \quad \int_{\mathcal{F}} u_0' \cdot \eta_i(y)dy + \omega\int_{\mathcal{G}} r_0'\eta_3(y)\cdot\eta_i(y)dS = 0, \quad i = 1,2,3.$$

Now, we define $u_1(x,t), q_1(x,t), r_1(x,t)$ as a solution of a linear problem

$$u_{1t} + 2\omega(e_3\times u_1) - \nu\nabla^2 u_1 + \nabla q_1 = 0, \quad \nabla\cdot u_1 = 0, \qquad x\in\mathcal{F},$$

$$T(u_1,q_1)N = -NB_0 r_1,$$

$$r_{1t} = u_1\cdot N - \frac{h_0(x)}{\int_{\mathcal{G}} h_0^2(y)dS}\int_{\mathcal{G}} h_0(y)u_1\cdot N(y)dS, \quad x\in\mathcal{G},$$

$$u_1(x,0) = u_0'(x), \quad x\in\mathcal{F}, \quad r_1(x,0) = r_0'(x), \quad x\in\mathcal{G};$$

then $u - u_1 = u_2$, $q - q_1 = q_2$, $\widetilde{\rho} - r_1 = \rho_2$ satisfy the relations

$$u_{2t} + 2\omega(e_3\times u_2) - \nu\nabla^2 u_2 + \nabla q_2 = f(u_1+u_2, q_1+q_2, r_1+\rho_2),$$

$$\nabla\cdot u_2 = 0, \qquad x\in\mathcal{F},$$

$$\Pi_0 S(u_2)N = b(u_1+u_2, r_1+\rho_2), \tag{4.6}$$

$$-q_2 + N\cdot S(u_2)N(x) + B_0\rho_2 = d(u_1+u_2, r_1+\rho_2),$$

$$r_{2t} = u_2\cdot N - \frac{h_0(x)}{\int_{\mathcal{G}} h_0^2(y)dS}\int_{\mathcal{G}} h_0(y)u_2\cdot N(y)dS + g(u_1+u_2, r_1+\rho_2),$$

$$\rho_2(x,0) = (\rho_0 - \varphi(\rho_0)) + r_0''(x), \quad u_2(x,0) = u_0''(x).$$

For the last problem an analogue of proposition 2.3 in [5] holds.

Proposition 4.1. *Given arbitrary $T > 0$, there exists a number $\epsilon_2(T) > 0$ such that in the case*

$$|u_0|_{C^{2+\alpha}(\mathcal{F})} + |\rho_0|_{C^{3+\alpha}(\mathcal{G})} \le \epsilon_2$$

problem (4.6) is uniquely solvable in the interval of time $t \in [0, T]$, and the solution satisfies the inequality

$$\sup_{\tau \le t} |u_{2t}(\cdot, \tau)|_{C^{\alpha}(\mathcal{F})} + |u_2(\cdot, \tau)|_{C^{2+\alpha}(\mathcal{F})} + \sup_{\tau \le t} |q_2(\cdot, \tau)|_{C^{1+\alpha}(\mathcal{F})} + |\rho_2(\cdot, t)|_{C^{3+\alpha}(\mathcal{G})}$$

$$\le c\Big(|u_2(\cdot, 0)|_{C^{2+\alpha}(\mathcal{F})} + |\rho_2(\cdot, 0)|_{C^{3+\alpha}(\mathcal{G})}\Big) \le c\Big(|w_0|_{C^{2+\alpha}(\mathcal{F})} + |\rho_0|_{C^{3+\alpha}(\mathcal{G})}\Big)^2. \quad (4.7)$$

Hence, the solution of (3.12) has the form

$$u = u_1 + u_2, \quad \widetilde{\rho} = r_1 + \rho_2.$$

Since $(u_1, r_1) = e^{tA}(u_0', r_0')$ can grow exponentially for appropriate choice of initial data, and (u_2, ρ_2) is controlled by (4.7), it is possible to show that $(u, \widetilde{\rho})$ must leave sooner or later a certain ball

$$|u(\cdot, t)|_{C^{2+\alpha}(\mathcal{F})} + |\widetilde{\rho}(\cdot, t)|_{C^{3+\alpha}(\mathcal{G})} = \epsilon$$

in the space $X = C^{2+\alpha}(\mathcal{F}) \times C^{3+\alpha}(\mathcal{G})$. Technically it is done exactly as in [5], and it is not necessary to reproduce the proof here.

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