CASCADE OF PHASE SHIFTS FOR NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider a semi-classical nonlinear Schrödinger equation. For initial data causing focusing at one point in the linear case, we study a nonlinearity which is super-critical in terms of asymptotic effects near the caustic. We prove the existence of infinitely many phase shifts appearing at the approach of the critical time. This phenomenon is suggested by a formal computation. The rigorous proof shows a quantitatively different asymptotic behavior. We explain these aspects, and discuss some problems left open.

1. Introduction

We consider the semi-classical limit of the Cauchy problem, for $(t,x)\in\mathbb{R}_+\times\mathbb{R}^n$:

$$i\varepsilon\partial_t \mathbf{u}^\varepsilon+\frac{\varepsilon^2}{2}\Delta \mathbf{u}^\varepsilon = f\left(|\mathbf{u}^\varepsilon|^2\right)\mathbf{u}^\varepsilon \quad ; \quad \mathbf{u}^\varepsilon_{|t=0}=\varepsilon^{k/2}a_0(x)e^{-i\frac{|x|^2}{2\varepsilon}}. \tag{1.1}$$

In the linear case $f\equiv 0$, the quadratic oscillations of the initial data cause focusing at the origin at time $t=1$ in the limit $\varepsilon\to 0$ (see Section 2.1). In the nonlinear case, the effective nonlinear effects strongly depend on the size of the initial data, that is on k. Changing notations, we consider:

$$i\varepsilon\partial_t u^\varepsilon+\frac{\varepsilon^2}{2}\Delta u^\varepsilon = f\left(\varepsilon^k|u^\varepsilon|^2\right)u^\varepsilon \quad ; \quad u^\varepsilon_{|t=0}=a_0(x)e^{-i\frac{|x|^2}{2\varepsilon}}. \tag{1.2}$$

In [1], we justified the general heuristics presented in [9], in the case of (1.2), for f homogeneous of degree σ, $f(y)=y^\sigma$. Two notions of criticality exist for k: outside the focal point, and near the focal point, where the amplitude of u^ε is strongly modified. We described the sub-critical and critical cases. The aim of the present paper is to study a supercritical case.

Consider the case $f(y)=y^\sigma$, and denote $\alpha=k\sigma$. If $a_0\in H^1(\mathbb{R}^n)$ with $|x|a_0\in L^2(\mathbb{R}^n)$ and $\sigma<2/(n-2)$ when $n\geq 3$, u^ε is defined globally in time in $H^1(\mathbb{R}^n)$. The following distinctions were established in [1]:

	$\alpha>n\sigma$	$\alpha=n\sigma$
$\alpha>1$	linear caustic linear WKB	nonlinear caustic linear WKB
$\alpha=1$	linear caustic nonlinear WKB	nonlinear caustic nonlinear WKB

The term "linear WKB" means that outside the caustic, the propagation of u^ε can be described by a geometrical optics approximation, with only linear effects involved at leading order. The term "linear caustic" means that nonlinear effects are negligible at leading order when the solution crosses the focal point. In either of the two critical cases, nonlinear phenomena are described (the doubly critical

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case was studied more precisely in [2]); we recall the case "nonlinear caustic, linear WKB" in Section 2.2. The case we study in this paper corresponds to $n\sigma > \alpha > 1$: super-critical caustic with linear WKB régime.

A look at conservation laws suggest the existence of new relevant scales. In the case discussed so far, the conservations of charge and energy read (see e.g. [3]):

$$\begin{aligned}
&\|u^\varepsilon(t)\|_{L^2} = \|a_0\|_{L^2}\,,\\
&\frac{1}{2}\|\varepsilon\nabla_x u^\varepsilon(t)\|_{L^2}^2 + \frac{\varepsilon^\alpha}{\sigma+1}\|u^\varepsilon(t)\|_{L^{2\sigma+2}}^{2\sigma+2} = \text{const.} = \mathcal{O}(1) \underset{\varepsilon\to 0}{\sim} \frac{1}{2}\|xa_0\|_{L^2}^2\,.
\end{aligned} \tag{1.3}$$

When $\alpha \geq n\sigma$, the boundedness of u^ε and $\varepsilon\nabla_x u^\varepsilon$ in L^2 implies, along with Gagliardo–Nirenberg inequalities:

$$\varepsilon^\alpha\|u^\varepsilon(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \leq \varepsilon^{n\sigma}\|u^\varepsilon(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \lesssim \|u^\varepsilon(t)\|_{L^2}^{2-(n-2)\sigma}\|\varepsilon\nabla_x u^\varepsilon(t)\|_{L^2}^{n\sigma} = \mathcal{O}(1)\,.$$

Thus, linear arguments allow us to recover a control of the nonlinear term in the energy. Such a line of reasoning fails when $\alpha < n\sigma$: the control provided by the conservation of energy hides stronger nonlinear effects. In the linear case $f \equiv 0$, and $a_0 \in \mathcal{S}(\mathbb{R}^n)$, one can check that the following point-wise estimate holds:

$$|u^\varepsilon(t,x)| \lesssim \frac{1}{(\varepsilon + |t-1|)^{n/2}}\,. \tag{1.4}$$

In the four cases of the table, the same estimate holds for the nonlinear solution in space dimension one ([1, 2]); like in the linear case, it is sharp. Thus, the above estimate $\varepsilon^{n\sigma}\|u^\varepsilon(t)\|_{L^{2\sigma+2}}^{2\sigma+2} = \mathcal{O}(1)$ is sharp only near the focal point. We now use the *a priori* estimate $\varepsilon^{\alpha}\|u^\varepsilon(t)\|_{L^{2\sigma+2}}^{2\sigma+2} = \mathcal{O}(1)$ given by the conservation of energy only for $t \approx 1$. Assuming like in all the cases of the above table that at time $t=1$, u^ε is described by a concentrating profile,

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \frac{1}{\varepsilon^{n\gamma/2}}\phi\left(\frac{x}{\varepsilon^\gamma}\right)\,,$$

we check that the "linear" value $\gamma = 1$ is forbidden (the power of ε in front of ϕ is to ensure the L^2-norm conservation). Guessing that the nonlinear term $\varepsilon^{\alpha}\|u^\varepsilon(t)\|_{L^{2\sigma+2}}^{2\sigma+2}$ in the energy is exactly of order $\mathcal{O}(1)$ at the caustic, we find $\gamma = \alpha/(n\sigma)$, that is:

$$\gamma = \frac{k}{n} < 1\,. \tag{1.5}$$

We will not prove that the above argument is correct (see Section 6), but we will show that the scale ε^γ is an important feature of this problem. Notice also that the above argument suggests that the amplification of the solution u^ε as time goes to 1 is less important than in the linear case; super-critical phenomena may occur in the phase, and also affect the amplitude.

We now go back to the notation (1.2), and do not assume in general that the nonlinearity is homogeneous (unless it is cubic):

Assumptions 1.1. *The space dimension is $n \geq 2$.*
The initial amplitude belongs to the Schwartz space: $a_0 \in \mathcal{S}(\mathbb{R}^n)$.
The nonlinearity is smooth: $f \in C^\infty(\mathbb{R}_+;\mathbb{R})$.
$f(0) = 0$ and $f' > 0$. In particular, the nonlinearity is cubic at the origin.

Remark 1.2. We suppose $a_0 \in \mathcal{S}(\mathbb{R}^n)$ to avoid to count derivatives when not necessary. We could as well assume that a_0 belongs to Sobolev type spaces. If we require a control on the growth of f at infinity, $(0 \leq) f(y) \lesssim \langle y\rangle^q$ for $q < \frac{2}{n-2}$ when $n \geq 3$, then for every fixed $\varepsilon > 0$, u^ε is global in time, continuous with values in $H^1(\mathbb{R}^n)$ (see e.g. [3]). This includes a cubic nonlinearity in space dimension two or three.

Remark 1.3. The assumption $f(0) = 0$ is only to simplify notations, since replacing f with $f - f(0)$ turns $u^\varepsilon(t,x)$ into $u^\varepsilon(t,x)e^{if(0)t/\varepsilon}$.

Remark 1.4. The assumption of the nonlinearity being cubic at the origin is reminiscent of the paper by E. Grenier [8] (see also P. Gérard [7]). The proof of our main result relies on ideas introduced in [8] (see Section 4).

Remark 1.5. The one-dimensional cubic nonlinear Schrödinger equation is integrable. The case $k=0$ with more general WKB data was treated in [10].

Before stating our main result, we give the following definition (see e.g. [13]):

Definition 1.6. If $T>0$, $(k_j)_{j\geq 1}$ is an increasing sequence of real numbers, $(\phi_j)_{j\geq 1}$ is a sequence in $H^\infty(\mathbb{R}^n):=\cap_{s\geq 0}H^s(\mathbb{R}^n)$, and $\phi\in C([0,T];H^s(\mathbb{R}^n))$ for every $s>0$, the asymptotic relation

$$\phi(t,x)\sim \sum_{j\geq 1} t^{k_J}\phi_j(x) \quad \text{as } t\to 0$$

means that for every integer $J\geq 1$ and every $s>0$,

$$\left\| \phi(t,\cdot)-\sum_{j=1}^J t^{k_j}\phi_j\right\|_{H^s(\mathbb{R}^n)} = o\left(t^{k_J}\right) \quad \text{as } t\to 0\,.$$

Theorem 1.7. *Let Assumptions 1.1 be satisfied. Assume $n>k>1$. Then there exists $T>0$ independent of $\varepsilon\in]0,1]$, a sequence $(\phi_j)_{j\geq 1}$ in $H^\infty(\mathbb{R}^n)$, and $\phi\in C([0,T];H^s(\mathbb{R}^n))$ for every $s>0$, such that:*
1. *$\phi(t,x)\sim\sum_{j\geq 1}t^{jn-1}\phi_j(x)$ as $t\to 0$.*
2. *For $1-t\gg\varepsilon^\gamma$ ($\gamma=k/n<1$), the asymptotic behavior of u^ε is given by:*

$$\limsup_{\varepsilon\to 0}\sup_{0\leq t\leq 1-\Lambda\varepsilon^\gamma}\|u^\varepsilon(t)-v^\varepsilon(t)\|_{L^2(\mathbb{R}^n)} \underset{\Lambda\to+\infty}{\longrightarrow} 0\,,$$

$$\text{where}\quad v^\varepsilon(t,x)=\frac{e^{i\frac{|x|^2}{2\varepsilon(t-1)}}}{(1-t)^{n/2}}a_0\left(\frac{x}{1-t}\right)\exp\left(i\varepsilon^{\gamma-1}\phi\left(\frac{\varepsilon^\gamma}{1-t},\frac{x}{1-t}\right)\right).$$

We now comment this result. In the linear case $f\equiv 0$, the above result holds with $\gamma=1$ and $\phi\equiv 0$ (see Section 2.1). We recall in Section 2.2 that in the critical case "nonlinear caustic, linear WKB", the same asymptotic as in the linear case holds for $1-t\gg\varepsilon$. The case $k<n$ is super-critical as far as nonlinear effects near $t=1$ are concerned. We emphasize two important features in the above result: the analysis stops sooner than $1-t\gg\varepsilon$, and nonlinear effects cause the presence of the (nontrivial) phase ϕ. For $1-t\gg\varepsilon^\gamma$, we have

$$\varepsilon^{\gamma-1}\phi\left(\frac{\varepsilon^\gamma}{1-t},\frac{x}{1-t}\right)\sim\sum_{j\geq 1}\frac{\varepsilon^{jk-1}}{(1-t)^{jn-1}}\phi_j\left(\frac{x}{1-t}\right)\,.$$

The above phase shift starts being relevant for $1-t\sim\varepsilon^{\frac{k-1}{n-1}}$ (recall that $n>k>1$); this is the first boundary layer where nonlinear effects appear at leading order, measured by ϕ_1. We will check that this phase shift is relevant: ϕ_1 is not zero in general, see (5.1) below. We then have a countable number of boundary layers in time, of size

$$1-t\sim\varepsilon^{\frac{jk-1}{jn-1}}\,,$$

which reach the layer $1-t\sim\varepsilon^\gamma$ in the limit $j\to+\infty$. At each new boundary layer, a new phase ϕ_j becomes relevant at leading order. In general, none of the ϕ_j's is zero: see e.g. (5.3) for ϕ_2. The result of a cascade of phases can be compared to the one discovered recently by C. Cheverry [5] in the case of fluid dynamics, although the phenomenon seems to be different.

The assumption $k>1$ means that we start with a linear WKB régime. Indeed, for small positive time, u^ε remains of order $\mathcal{O}(1)$, and $f(\varepsilon^k|u^\varepsilon|^2)\sim\varepsilon^k|u^\varepsilon|^2f'(0)$.

The main term is then the same as in [1] with $\sigma = 1$ and $\alpha = k$. As recalled in the above table, $\alpha > 1$ corresponds to a propagation which is linear at leading order.

Each phase shift oscillates at a rate between $\mathcal{O}(1)$ (when it starts being relevant) and $\mathcal{O}(\varepsilon^{\gamma-1})$ (when it reaches the layer of size ε^γ). Since $\gamma > 0$, this means that each phase shift is rapidly oscillating at the scale of the amplitude, but oscillating strictly more slowly than the geometric phase $\frac{|x|^2}{2\varepsilon(t-1)}$, for $1 - t \gg \varepsilon^\gamma$. We will see in Section 6 that for $1 - t = \mathcal{O}(\varepsilon^\gamma)$, all the terms in ϕ, plus the geometric phase, have the same order: all these phases become comparable.

We will prove a more precise asymptotics than the L^2 estimate of Theorem 1.7: see Proposition 4.1 and (4.1). We restricted our attention to the L^2 norm for the sake of brevity.

Unfortunately, our analysis stops at the boundary layer of size ε^γ (we can only go up to $1 - t = \lambda\varepsilon^\gamma$ for some finite λ). We will discuss this fact in Section 6, and explain why we took care of never speaking of “focal point” in the super-critical case, but only of caustic (as a matter of fact, even the existence of a caustic is not clear, see Section 6). For instance, the geometry of the propagation is not known for $1 - t \leq \varepsilon^\gamma$, while the analysis shows that it occurs on the rays of linear geometric optics before this layer (see Figure 1). On the other hand, we know that the order

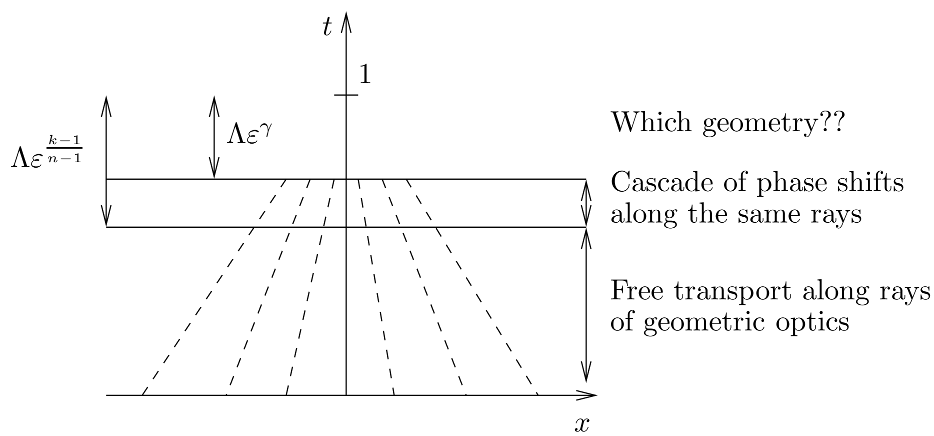

FIGURE 1. Geometry of the propagation, with morally $\Lambda \to +\infty$.

of magnitude of the amplitude changes in the boundary layer of size ε^γ. Recall that Theorem 1.7 describes the asymptotic behavior of u^ε for $1 - t \geq \Lambda\varepsilon^\gamma$, in the limit $\Lambda \to +\infty$. In this region, leading order nonlinear effects are visible only in the phase. As mentioned above, our analysis is valid for $1 - t \geq \lambda\varepsilon^\gamma$, for some finite λ. Between the initial time and this region, the amplitude of u^ε varies like $(1-t)^{-n/2}$, and changes from $\mathcal{O}(1)$ to $\mathcal{O}(\varepsilon^{-k/2})$.

The rest of the paper is organized as follows. In Section 2, we recall the proof of the analog of Theorem 1.7 in the linear and critical nonlinear cases. In Section 3, we present a formal computation that suggests a result like Theorem 1.7. Based on the result by E. Grenier [8] and a “semi-classical conformal transform”, we give the proof of Theorem 1.7 in Section 4. In Section 5, we compare the rigorous approach with the formal result of Section 3. The discussion about the possible phenomena for $t \geq 1 - \lambda\varepsilon^\gamma$ appears in the final Section 6.

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2. Free and critical cases

2.1. The linear equation. Consider the linear equation:

$$i\varepsilon\partial_t u^\varepsilon_{\rm lin} + \frac{\varepsilon^2}{2}\Delta u^\varepsilon_{\rm lin} = 0\,, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n\ ; \quad u^\varepsilon_{{\rm lin}|t=0} = a_0(x)e^{-i\frac{|x|^2}{2\varepsilon}}. \tag{2.1}$$

As $\varepsilon \to 0$, the rays of geometric optics (classical trajectories) are lines $\frac{x}{1-t} = \text{const.}$, and meet at the origin at time $t = 1$. Indeed, the bicharacteristic curves are defined by the Hamilton flow associated to $p(t,x,\tau,\xi) = \tau + \frac{|\xi|^2}{2}$:

$$\dot{t} = 1 \quad ; \quad \dot{x} = \xi \quad ; \quad \dot{\tau} = \dot{\xi} = 0 \quad ; \quad x(0) = x_0 \quad ; \quad \xi(0) = \nabla\phi(0,x(0)) = -x_0\,.$$

Of course, $u^\varepsilon_{\rm lin}$ can be expressed in terms of an oscillatory integral:

$$u^\varepsilon(t,x) = \frac{1}{(2i\pi\varepsilon t)^{n/2}}\int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{2\varepsilon t} - i\frac{|y|^2}{2\varepsilon}} a_0(y)dy\,. \tag{2.2}$$

Applying stationary phase formula yields the same result as using WKB methods below, up to the same boundary layer. Seek

$$u^\varepsilon_{\rm lin}(t,x) \underset{\varepsilon\to 0}{\sim} v^\varepsilon_{\rm lin}(t,x) = v^0(t,x)e^{i\frac{\phi(t,x)}{\varepsilon}}\,.$$

Plugging this into (2.1) and canceling the $\mathcal{O}(\varepsilon^0)$ and $\mathcal{O}(\varepsilon^1)$ terms, we find:

$$\partial_t\phi + \frac{1}{2}|\nabla_x\phi|^2 = 0\,, \quad \phi(0,x) = -\frac{|x|^2}{2}\ ;$$
$$\partial_t v^0 + \nabla_x\phi\cdot\nabla_x v^0 + \frac{1}{2}v^0\Delta\phi = 0\,, \quad v^0(0,x) = a_0(x)\,.$$

For $t < 1$, one has explicitly:

$$\phi(t,x) = \frac{|x|^2}{2(t-1)} \quad ; \quad v^0(t,x) = \frac{1}{(1-t)^{n/2}}a_0\left(\frac{x}{1-t}\right)\,.$$

Moreover, $v^\varepsilon_{\rm lin}$ solves:

$$i\varepsilon\partial_t v^\varepsilon_{\rm lin} + \frac{\varepsilon^2}{2}\Delta v^\varepsilon_{\rm lin} = \frac{\varepsilon^2}{2(1-t)^2}\frac{e^{i\frac{|x|^2}{2(t-1)}}}{(1-t)^{n/2}}\Delta a_0\left(\frac{x}{1-t}\right)\ ;\ v^\varepsilon_{{\rm lin}|t=0} = a_0(x)e^{-i\frac{|x|^2}{2\varepsilon}}.$$

Let $r^\varepsilon(t,x)$ denote the source term: $\|r^\varepsilon(t)\|_{L^2} \lesssim \dfrac{\varepsilon^2}{(1-t)^2}$. Standard energy estimates for Schrödinger equation yield:

$$\varepsilon\frac{d}{dt}\|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^2} \lesssim \|r^\varepsilon(t)\|_{L^2} \lesssim \frac{\varepsilon^2}{(1-t)^2}\,,$$

and

$$\sup_{0\le s\le t}\|u^\varepsilon(s) - v^\varepsilon(s)\|_{L^2} \lesssim \frac{1}{\varepsilon}\int_0^t \|r^\varepsilon(s)\|_{L^2}ds \lesssim \frac{\varepsilon}{1-t}\,.$$

Thus, WKB approximation is interesting up to a boundary layer in time of size ε before the focus. Moreover, $v^\varepsilon_{\rm lin}$ is exactly the approximate solution of Theorem 1.7 with $\phi \equiv 0$. Past this boundary layer, (2.2) shows that for $|1-t| = \mathcal{O}(\varepsilon)$,

$$u^\varepsilon(t,x) \sim \frac{e^{i\frac{|x|^2}{2\varepsilon}}}{(2i\pi\varepsilon)^{n/2}}\int e^{-i\frac{x\cdot y}{\varepsilon}}a_0(y)dy = \frac{e^{i\frac{|x|^2}{2\varepsilon}}}{\varepsilon^{n/2}}\mathcal{F}a_0\left(\frac{x}{\varepsilon}\right) \sim \frac{1}{\varepsilon^{n/2}}\mathcal{F}a_0\left(\frac{x}{\varepsilon}\right),$$

where $\mathcal{F}$ denotes the Fourier transform. For $t - 1 \gg \varepsilon$, stationary phase formula yields the same asymptotic description as above, up to the Maslov index (see [6, 1]). In particular, we see that (1.4) holds for $u^\varepsilon_{\rm lin}$, and is sharp.

2.2. **The critical nonlinear case.** We recall the main result of [1]. Consider (1.2) in the case $k = n$, which is critical concerning the role of the nonlinearity near the focal point. Introduce the scaling

$$u^\varepsilon(t,x) = \frac{1}{\varepsilon^{n/2}} \psi^\varepsilon \left(\frac{t-1}{\varepsilon}, \frac{x}{\varepsilon} \right).$$

Then the function ψ^ε solves

$$i\partial_t \psi^\varepsilon + \frac{1}{2}\Delta \psi^\varepsilon = f\left(|\psi^\varepsilon|^2\right)\psi^\varepsilon \quad ; \quad \psi^\varepsilon_{|t=\frac{-1}{\varepsilon}} = \varepsilon^{n/2} a_0(\varepsilon x) e^{-i\varepsilon \frac{|x|^2}{2}} . \tag{2.3}$$

A way to understand criticality is that ε has disappeared from the equation satisfied by ψ^ε. Using global well-posedness results for nonlinear Schrödinger equations (under assumptions on the nonlinearity which are different from Assumptions 1.1, see e.g. [3]), one has

$$\|\psi^\varepsilon - \psi\|_{L^\infty(\mathbb{R}; H^1(\mathbb{R}^n))} \to 0 \text{ as } \varepsilon \to 0 ,$$

where ψ is the (global) solution of the Cauchy problem

$$i\partial_t \psi + \frac{1}{2}\Delta \psi = f\left(|\psi|^2\right)\psi \quad ; \quad e^{-i\frac{t}{2}\Delta}\psi(t,x)\big|_{t=-\infty} = \mathcal{F}^{-1}(a_0)(x) . \tag{2.4}$$

Scattering theory shows two interesting features: for large $|t|$, $\psi(t,x)$ behaves like a solution of the linear Schrödinger equation. This implies that for $1-t \gg \varepsilon$, the solution u^ε can be approximated by $u^\varepsilon_{\rm lin}$ (or $v^\varepsilon_{\rm lin}$): no nonlinear effect is relevant before the same boundary layer as before. The second point is that for $|1-t| \lesssim \varepsilon$, nonlinear effects occur at leading order, and are measured (in average) by the nonlinear scattering operator associated to (2.4).

3. A formal computation

From now on, we assume that $k < n$. To simplify notations, and since the Assumptions 1.1 will be needed for rigorous proofs only, consider the case of an homogeneous nonlinearity: $f(y) = y^\sigma$, and denote $\alpha = k\sigma$. Then (1.2) becomes

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon^\alpha |u^\varepsilon|^{2\sigma} u^\varepsilon \quad ; \quad u^\varepsilon_{|t=0} = a_0(x) e^{-i\frac{|x|^2}{2\varepsilon}} . \tag{3.1}$$

The caustic is supercritical: $n\sigma > \alpha$. We also assume $\alpha > 1$ (linear WKB). Because this section is only formal, we shall be very brief about the computations, and only give the main steps.

3.1. **A first boundary layer.** Two approaches (at least) lead to the same result: Lagrangian integral with stationary phase formula (like in [2] where the critical one-dimensional cubic case is considered), and generalized WKB methods. We shall retain the second one, which we use in the next subsection. Seek

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} v_1^\varepsilon(t,x) = u^0(t,x) e^{i\frac{\phi(t,x)}{\varepsilon}} ,$$

and change the usual hierarchy to force the contribution of the nonlinear term to appear in the transport equation:

$$\partial_t \phi + \frac{1}{2}|\nabla_x \phi|^2 = 0 , \quad \phi(0,x) = -\frac{|x|^2}{2} ;$$
$$\partial_t u^0 + \nabla_x \phi \cdot \nabla_x u^0 + \frac{1}{2} u^0 \Delta \phi = -i\varepsilon^{\alpha-1}|u^0|^{2\sigma} u^0 , \quad u^0(0,x) = a_0(x) .$$

The eikonal equation is the same as in Section 2.1, as well as its solution. The transport equation is an ordinary differential equation along the rays of geometric optics $\frac{x}{1-t} = \text{const.}$, of the form

$$\dot{y} = -i\varepsilon^{\alpha-1}|y|^{2\sigma} y .$$

The modulus of u^0 is constant along rays, and

$$u^0(t,x) = \frac{1}{(1-t)^{n/2}} a_0\left(\frac{x}{1-t}\right) \exp\left(-i\varepsilon^{\alpha-1}\left|a_0\left(\frac{x}{1-t}\right)\right|^{2\sigma}\int_0^t \frac{ds}{(1-s)^{n\sigma}}\right).$$

Note that the notation is no longer relevant, since u^0 now depends on ε. We have a new boundary layer in time, of size ε^β before the focus, where

$$\beta = \frac{\alpha-1}{n\sigma-1}\,.$$

For $1-t\sim\varepsilon^\beta$, the above phase shift measures relevant nonlinear effects. We have:

$$i\varepsilon\partial_t v_1^\varepsilon + \frac{\varepsilon^2}{2}\Delta v_1^\varepsilon = \varepsilon^\alpha |v_1^\varepsilon|^{2\sigma} v_1^\varepsilon + r_1^\varepsilon \quad ; \quad v^\varepsilon_{1|t=0} = a_0(x) e^{-i\frac{|x|^2}{2\varepsilon}}\,,$$

with

$$\frac{1}{\varepsilon}\int_0^t \|r_1^\varepsilon(s)\|_{L^2} ds \lesssim \frac{\varepsilon}{1-t} + \frac{\varepsilon^{2\alpha-1}}{(1-t)^{2n\sigma-1}}\,.$$

Following the energy estimates of Section 2.1, this quantity might be the one that dictates the size of the error $u^\varepsilon - v_1^\varepsilon$ (see Section 5 for a discussion on that issue). The second term is "new" (the first term is the same as in Section 2.1), and suggests the existence of a second boundary layer, of size $\varepsilon^{\frac{2\alpha-1}{2n\sigma-1}}$.

3.2. **Infinitely many boundary layers: cascade of phase shifts.** Seek an approximate solution of the form:

$$v^\varepsilon(t,x) = \frac{1}{(1-t)^{n/2}} a_0\left(\frac{x}{1-t}\right) e^{i\phi^\varepsilon(t,x)}, \quad \phi^\varepsilon(t,x) = \frac{|x|^2}{2\varepsilon(t-1)} + g^\varepsilon(t,x)\,.$$

We find

$$i\varepsilon\partial_t v^\varepsilon + \frac{\varepsilon^2}{2}\Delta v^\varepsilon = \left(i\frac{\varepsilon^2}{2}\Delta g^\varepsilon - \varepsilon\partial_t g^\varepsilon - \frac{\varepsilon^2}{2}|\nabla_x g^\varepsilon|^2 + \frac{\varepsilon}{1-t} x\cdot\nabla_x g^\varepsilon\right) v^\varepsilon$$
$$+ i\frac{\varepsilon^2}{(1-t)^{\frac{n}{2}+1}}\nabla_x g^\varepsilon\cdot\nabla a_0\left(\frac{x}{1-t}\right) e^{i\phi^\varepsilon} + \frac{1}{2}\left(\frac{\varepsilon}{1-t}\right)^2 \frac{e^{i\phi^\varepsilon}}{(1-t)^{n/2}}\Delta a_0\left(\frac{x}{1-t}\right).$$

As suggested by the previous paragraph, write

$$g^\varepsilon(t,x) = \frac{1}{\varepsilon}\int_0^t h\left(\frac{\varepsilon^\alpha}{(1-s)^{n\sigma}},\frac{x}{1-t}\right) ds\,, \quad \text{with } h(z,\xi) \sim \sum_{j\geq 1} z^j g_j(\xi)\,. \tag{3.2}$$

In the equation solved by v^ε, the last term is the "same" as in the linear case: it becomes relevant only in a boundary layer of size ε. Since our approach will lead us to the boundary layer of size ε^γ (recall that $\gamma = k/n = \alpha/n\sigma < 1$), we ignore that term.

The remaining terms with a factor i are of order, in L^2,

$$\varepsilon^2\|\Delta g^\varepsilon(t)\|_{L^\infty} + \frac{\varepsilon^2}{1-t}\|\nabla_x g^\varepsilon(t)\|_{L^\infty} \lesssim \frac{\varepsilon}{(1-t)^2}\int_{t_0}^t \frac{\varepsilon^\alpha}{(1-s)^{n\sigma}} ds \lesssim \frac{\varepsilon^{\alpha+1}}{(1-t)^{n\sigma+1}}\,,$$

and their contribution is also left out in this computation.

Now we require that v^ε be an approximate solution to (3.1):

$$\left(\partial_t - \frac{x}{1-t}\cdot\nabla_x\right) g^\varepsilon + \frac{\varepsilon}{2}|\nabla_x g^\varepsilon|^2 = -\frac{\varepsilon^{\alpha-1}}{(1-t)^{n\sigma}}\left|a_0\left(\frac{x}{1-t}\right)\right|^{2\sigma}. \tag{3.3}$$

Using (3.2), we get:

$$g_1(\xi) = -|a_0(\xi)|^{2\sigma}\,,$$
$$\text{for } j\geq 2, \quad g_j(\xi) = -\frac{1}{2}\sum_{p+q=j}\frac{1}{(pn\sigma-1)(qn\sigma-1)}\nabla g_p\cdot\nabla g_q\,, \tag{3.4}$$

with the convention $g_0 \equiv 0$. This algorithm produces smooth solutions provided that $|a_0(\xi)|^{2\sigma}$ is smooth ($\sigma \in \mathbb{N}^*$ or a_0 Gaussian for instance). We neglected the terms corresponding to $s=0$ in the integration (3.2): this does not increase the error, since $n\sigma > \alpha > 1$. Defining

$$\begin{aligned}
\widetilde{g}_N^\varepsilon(t,x) &= \frac{1}{\varepsilon}\sum_{j=1}^N \int_0^t \left(\frac{\varepsilon^\gamma}{1-s}\right)^{n\sigma j} ds \times g_j\left(\frac{x}{1-t}\right),\\
\widetilde{v}_N^\varepsilon(t,x) &= \frac{1}{(1-t)^{n/2}} a_0\left(\frac{x}{1-t}\right) e^{i\frac{|x|^2}{2\varepsilon(t-1)}+\widetilde{g}_N^\varepsilon(t,x)},
\end{aligned}$$

the approximate solution $\widetilde{v}_N^\varepsilon$ solves

$$i\varepsilon\partial_t \widetilde{v}_N^\varepsilon + \frac{\varepsilon^2}{2}\Delta \widetilde{v}_N^\varepsilon = \varepsilon^\alpha |\widetilde{v}_N^\varepsilon|^{2\sigma}\widetilde{v}_N^\varepsilon + \widetilde{r}_N^\varepsilon \quad ; \quad \widetilde{v}^\varepsilon_{N|t=0} = a_0(x) e^{-i\frac{|x|^2}{2\varepsilon}},$$

with, for $1-t \geq \varepsilon^\gamma$:

$$\frac{1}{\varepsilon}\int_0^t \|\widetilde{r}_N^\varepsilon(s)\|_{L^2} ds \lesssim \frac{\varepsilon^{(N+1)\alpha-1}}{(1-t)^{(N+1)n\sigma-1}} + \frac{\varepsilon^\alpha}{(1-t)^{n\sigma}}.$$

To compare with Theorem 1.7, remove the terms corresponding to $s=0$ in the integration (recall that $n\sigma>1$), and define:

$$\begin{aligned}
g_N^\varepsilon(t,x) &= \frac{1}{\varepsilon}\sum_{j=1}^N \int_{-\infty}^t \left(\frac{\varepsilon^\gamma}{1-s}\right)^{n\sigma j} ds \times g_j\left(\frac{x}{1-t}\right),\\
v_N^\varepsilon(t,x) &= \frac{1}{(1-t)^{n/2}} a_0\left(\frac{x}{1-t}\right) e^{i\frac{|x|^2}{2\varepsilon(t-1)}+g_N^\varepsilon(t,x)}.
\end{aligned}$$

By definition, we have $\|v_N^\varepsilon(t) - \widetilde{v}_N^\varepsilon(t)\|_{L^2} = \mathcal{O}(\varepsilon^{\alpha-1})$ for $1-t\geq\varepsilon^\gamma$. One can check that v_N^ε solves

$$i\varepsilon\partial_t v_N^\varepsilon + \frac{\varepsilon^2}{2}\Delta v_N^\varepsilon = \varepsilon^\alpha |v_N^\varepsilon|^{2\sigma} v_N^\varepsilon + r_N^\varepsilon,$$

with

$$\begin{aligned}
r_N^\varepsilon(t,x) = &\left(\frac{\varepsilon^\alpha}{(1-t)^{n\sigma}}\right)^{N+1} v_N^\varepsilon(t,x) g_{N+1}\left(\frac{x}{1-t}\right)\\
&+\frac{1}{2}\left(\frac{\varepsilon}{1-t}\right)^2 e^{ig_N^\varepsilon(t,x)} \frac{e^{i\frac{|x|^2}{2\varepsilon(t-1)}}}{(1-t)^{n/2}}\Delta a_0\left(\frac{x}{1-t}\right)\\
&+i\frac{\varepsilon}{1-t}\sum_{j=1}^N \frac{1}{jn\sigma-1}\left(\frac{\varepsilon^\alpha}{(1-t)^{n\sigma}}\right)^j \Bigg(v_N^\varepsilon(t,x)\Delta g_j\left(\frac{x}{1-t}\right)\\
&\qquad + e^{ig_N^\varepsilon(t,x)}\frac{e^{i\frac{|x|^2}{2\varepsilon(t-1)}}}{(1-t)^{n/2}}\nabla g_j\cdot\nabla a_0\left(\frac{x}{1-t}\right)\Bigg).
\end{aligned} \tag{3.5}$$

We have the following result:

Proposition 3.1 (Formal approximation to (3.1))**.** *Let $n\sigma>\alpha>1$, $a_0\in\mathcal{S}(\mathbb{R}^n)$, and fix $N\in\mathbb{N}^*$. Denote*

$$g_N^\varepsilon(t,x) = \sum_{j=1}^N \frac{\varepsilon^{j\alpha-1}}{(1-t)^{jn\sigma-1}}\frac{1}{jn\sigma-1} g_j\left(\frac{x}{1-t}\right),$$

and let v_N^ε be the associated approximate solution. The function v_N^ε solves

$$i\varepsilon\partial_t v_N^\varepsilon + \frac{\varepsilon^2}{2}\Delta v_N^\varepsilon = \varepsilon^\alpha |v_N^\varepsilon|^{2\sigma} v_N^\varepsilon + r_N^\varepsilon \ ; \ v^\varepsilon_{N|t=0} = a_0(x)e^{-i\frac{|x|^2}{2\varepsilon}} + \mathcal{O}\left(\varepsilon^{\alpha-1}\right) \ \text{in } L^2.$$

For $1-t \geq \varepsilon^\gamma = \varepsilon^{\frac{\alpha}{n\sigma}}$*, the source term satisfies:*

$$\frac{1}{\varepsilon}\int_0^t \|r_N^\varepsilon(s)\|_{L^2} ds \lesssim \frac{\varepsilon^{(N+1)\alpha-1}}{(1-t)^{(N+1)n\sigma-1}} + \frac{\varepsilon^\alpha}{(1-t)^{n\sigma}} .$$

For $1 \leq j \leq N$, the $j^{\rm th}$ term of the series defining g_N^ε becomes relevant in a boundary layer of size $\varepsilon^{\frac{j\alpha-1}{nj\sigma-1}}$: in the limit $N \to +\infty$, a countable family of boundary layers appear, between ε^β and ε^γ. In the case $\sigma = 1$, which is the only homogeneous nonlinearity consistent with Assumptions 1.1, we have $\alpha = k$ and we find the boundary layers announced in the introduction.

Letting $N \to +\infty$ (using Borel lemma, see e.g. [13]), we find:

$$\frac{1}{\varepsilon}\int_0^t \|r^\varepsilon(s)\|_{L^2} ds \lesssim \frac{\varepsilon^\alpha}{(1-t)^{n\sigma}} ,$$

which is small for $1-t \gg \varepsilon^\gamma$.

Remark 3.2. In the critical case $\alpha = n\sigma > 1$, we have $\beta = \gamma = 1$: the above boundary layers "collapse" one on another. There are no such phase shifts as above.

We point out that the sole estimate of the source term proves nothing. In a stability argument, the nonlinearity $|u^\varepsilon|^{2\sigma}u^\varepsilon - |v_N^\varepsilon|^{2\sigma}v_N^\varepsilon$ is usually treated by a Gronwall type argument. If the nonlinearity is "too strong", then the above estimate, which is completely relevant in the linear case, does not necessarily account for the size of the error. Since we are in a super-critical case, it is not surprising that Proposition 3.1 is only a formal result. This remark can be compared to the approach in [7]. To justify a WKB expansion for the nonlinear equation

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = f\left(|u^\varepsilon|^2\right)u^\varepsilon ,$$

constructing an approximate solution that solves

$$i\varepsilon\partial_t u^\varepsilon_{\rm app} + \frac{\varepsilon^2}{2}\Delta u^\varepsilon_{\rm app} = f\left(|u^\varepsilon_{\rm app}|^2\right)u^\varepsilon_{\rm app} + \mathcal{O}\left(\varepsilon^\infty\right) ,$$

is not sufficient. Indeed, the computations in [7] show that energy estimates and Gronwall lemma do not yield better than

$$\left\|u^\varepsilon(t) - u^\varepsilon_{\rm app}(t)\right\|_{L^2} \leq e^{Ct/\varepsilon}\mathcal{O}\left(\varepsilon^\infty\right).$$

This is the reason why in [7], WKB expansions are justified for analytic data. This assumption yields a source term for $u^\varepsilon_{\rm app}$ which is $\mathcal{O}(e^{-\delta/\varepsilon})$, counterbalancing the exponential growth of Gronwall lemma to lead to a good approximation on $[0,T]$ for $T>0$ independent of ε.

4. Rigorous results

We now prove Theorem 1.7. We will see that the approximate solution we find diverges from the one constructed above, a fact which we discuss in Section 5.

4.1. **Semi-classical conformal transform.** Introduce the new unknown function ψ^ε given by:

$$u^\varepsilon(t,x) = \frac{1}{(1-t)^{n/2}}\psi^\varepsilon\left(\frac{\varepsilon^\gamma}{1-t},\frac{x}{1-t}\right)e^{i\frac{|x|^2}{2\varepsilon(t-1)}} . \tag{4.1}$$

Recalling that $\gamma = \frac{k}{n} < 1$, denote

$$\hbar = \varepsilon^{1-\gamma} \to 0 . \tag{4.2}$$

Changing the notation $\psi^\varepsilon(\tau,\xi)$ into $\psi^\hbar(t,x)$, we check that (1.2) becomes:

$$i\hbar\partial_t\psi^\hbar+\frac{\hbar^2}{2}\Delta\psi^\hbar=t^{-2}f\left(t^n|\psi^\hbar|^2\right)\psi^\hbar \quad ; \quad \psi^\hbar\big|_{t=\hbar^{\frac{\gamma}{1-\gamma}}}=a_0(x)\,. \tag{4.3}$$

The singular term t^{-2} in factor of the nonlinearity is actually harmless: as t goes to zero, $t^{-2}f\left(t^n|\psi^\hbar|^2\right)\approx t^{n-2}|\psi^\hbar|^2f'(0)$, and is bounded since $n\geq 2$.

The proof of Theorem 1.7 is now reduced to the asymptotic expansion for $\psi^\hbar$ as $\hbar\to 0$ for $t\in[\hbar^{\frac{\gamma}{1-\gamma}},\frac{1}{\Lambda}]$. Denote $t_0^\hbar=\hbar^{\frac{\gamma}{1-\gamma}}$. We shall prove the following:

Proposition 4.1. *Let Assumptions 1.1 be satisfied. Assume $n>k>1$, and let $s\in\mathbb{N}$. Then there exists $T>0$ independent of $\hbar\in]0,1]$ such that for $t\in[t_0^\hbar,T]$, the function $\psi^\hbar$ can be written as $\psi^\hbar(t,x)=a^\hbar(t,x)e^{i\phi^\hbar(t,x)/\hbar}$, with*

$$\left\|a^\hbar-a\right\|_{L^\infty([t_0^\hbar,T];H^s)}+\left\|\phi^\hbar-\phi\right\|_{L^\infty([t_0^\hbar,T];H^s)}\to 0 \quad \textit{as } \hbar\to 0\,,$$

where (a,ϕ) solves

$$\begin{aligned}\partial_t\phi+\frac{1}{2}|\nabla_x\phi|^2+t^{-2}f\left(t^n|a|^2\right)=0 \quad &; \quad \phi_{|t=0}=0\,,\\ \partial_t a+\nabla_x\phi\cdot\nabla_x a+\frac{1}{2}a\Delta\phi=0 \quad &; \quad a_{|t=0}=a_0\,.\end{aligned} \tag{4.4}$$

Moreover,

$$\limsup_{\hbar\to 0}\sup_{t_0^\hbar\leq t\leq\tau}\left\|\psi^\hbar(t,x)-a_0\left(x\right)e^{i\phi(t,x)/\hbar}\right\|_{H^s(\mathbb{R}^n)}\to 0 \quad \textit{as } \tau\to 0\,.$$

The second point of Theorem 1.7 follows from the above proposition, since the transform (4.1) is L^2 unitary (see Proposition 4.3 below for the asymptotic expansion of ϕ). We could also include not only derivatives in the above estimates, but also momenta. As announced in the introduction, we chose to leave out this refinement. Note that except for two aspects, Proposition 4.1 is nothing but rewriting Theorems 1.1 and 1.3 of [8]. In our case, time is present in the nonlinearity, and data for $\psi^\hbar$ are prescribed at time $t_0^\hbar$ (with $t_0^\hbar\to 0$ as $\hbar\to 0$) instead of time zero.

4.2. **Construction of solutions to** (4.3)**.** We recall the ideas introduced by E. Grenier [8], and show how to handle the presence of time in the nonlinearity. The main idea in [8] is to write the solution of a semi-classical nonlinear Schrödinger equation as a WKB solution, where not only the amplitude may depend on the small parameter, but also the phase. This changes the usual WKB hierarchy, and overcomes the difficulties pointed out in [7]. Seek the solution of (4.3) of the form

$$\psi^\hbar(t,x)=a^\hbar(t,x)e^{i\phi^\hbar(t,x)/\hbar},$$

with

$$\begin{aligned}\partial_t\phi^\hbar+\frac{1}{2}\left|\nabla\phi^\hbar\right|^2+t^{-2}f\left(t^n|a^\hbar|^2\right)=0 \quad &; \quad \phi^\hbar\big|_{t=t_0^\hbar}=0\,,\\ \partial_t a^\hbar+\nabla\phi^\hbar\cdot\nabla a^\hbar+\frac{1}{2}a^\hbar\Delta\phi^\hbar=i\frac{\hbar}{2}\Delta a^\hbar \quad &; \quad a^\hbar\big|_{t=t_0^\hbar}=a_0\,.\end{aligned} \tag{4.5}$$

Introducing the "velocity" $\mathtt{v}^\hbar=\nabla\phi^\hbar$, (4.5) yields

$$\begin{aligned}\partial_t\mathtt{v}^\hbar+\mathtt{v}^\hbar\cdot\nabla\mathtt{v}^\hbar+2t^{n-2}f'\left(t^n|a^\hbar|^2\right)\operatorname{Re}\left(\overline{a^\hbar}\nabla a^\hbar\right)=0 \quad &; \quad \mathtt{v}^\hbar\big|_{t=t_0^\hbar}=0\,,\\ \partial_t a^\hbar+\mathtt{v}^\hbar\cdot\nabla a^\hbar+\frac{1}{2}a^\hbar\operatorname{div}\mathtt{v}^\hbar=i\frac{\hbar}{2}\Delta a^\hbar \quad &; \quad a^\hbar\big|_{t=t_0^\hbar}=a_0\,.\end{aligned} \tag{4.6}$$

To force the initial time to be zero, introduce

$$\widetilde{\mathtt{v}}^\hbar(t,x)=\mathtt{v}^\hbar\left(t+t_0^\hbar,x\right) \quad ; \quad \widetilde{a}^\hbar(t,x)=a^\hbar\left(t+t_0^\hbar,x\right).$$

Then (4.6) becomes

$$\partial_t \widetilde{\mathtt{v}}^\hbar + \widetilde{\mathtt{v}}^\hbar \cdot \nabla \widetilde{\mathtt{v}}^\hbar + 2\left(t+t_0^\hbar\right)^{n-2} f'\left(\left(t+t_0^\hbar\right)^n |\widetilde{a}^\hbar|^2\right) \mathrm{Re}\left(\overline{\widetilde{a}^\hbar} \nabla \widetilde{a}^\hbar\right) = 0 \ ; \ \widetilde{\mathtt{v}}^\hbar\big|_{t=0} = 0,$$
$$\partial_t \widetilde{a}^\hbar + \widetilde{\mathtt{v}}^\hbar \cdot \nabla \widetilde{a}^\hbar + \frac{1}{2} \widetilde{a}^\hbar \operatorname{div} \widetilde{\mathtt{v}}^\hbar = i\frac{\hbar}{2} \Delta \widetilde{a}^\hbar \ ; \ \widetilde{a}^\hbar\big|_{t=0} = a_0.$$

Notice that if $n = 2$ and $f' =$ const. (2D cubic equation, which is conformally invariant), the above system is exactly the same as in [8].

Separate real and imaginary parts of $\widetilde{a}^\hbar$, $\widetilde{a}^\hbar = \widetilde{\mathtt{a}}^\hbar + i\widetilde{\mathtt{b}}^\hbar$. Then we have

$$\partial_t \mathbf{u}^\hbar + \sum_{j=1}^n A_j(\mathbf{u}^\hbar)\partial_j \mathbf{u}^\hbar = \frac{\hbar}{2} L \mathbf{u}^\hbar \,, \tag{4.7}$$

$$\text{with} \quad \mathbf{u}^\hbar = \begin{pmatrix} \widetilde{\mathtt{a}}^\hbar \\ \widetilde{\mathtt{b}}^\hbar \\ \widetilde{\mathtt{v}}_1^\hbar \\ \vdots \\ \widetilde{\mathtt{v}}_n^\hbar \end{pmatrix} \quad , \quad L = \begin{pmatrix} 0 & -\Delta & 0 & \dots & 0 \\ \Delta & 0 & 0 & \dots & 0 \\ 0 & 0 & & 0_{n\times n} & \end{pmatrix},$$

$$\text{and} \quad A(\mathbf{u},\xi) = \sum_{j=1}^n A_j(\mathbf{u})\xi_j = \begin{pmatrix} \widetilde{\mathtt{v}}\cdot\xi & 0 & \frac{\widetilde{\mathtt{a}}}{2}\,{}^t\xi \\ 0 & \widetilde{\mathtt{v}}\cdot\xi & \frac{\widetilde{\mathtt{b}}}{2}\,{}^t\xi \\ 2\left(t+t_0^\hbar\right)^{n-2} f'\widetilde{\mathtt{a}}\,\xi & 2\left(t+t_0^\hbar\right)^{n-2} f'\widetilde{\mathtt{b}}\,\xi & \widetilde{\mathtt{v}}\cdot\xi I_n \end{pmatrix},$$

where f' stands for $f'\left(\left(t+t_0^\hbar\right)^n (|\widetilde{\mathtt{a}}|^2 + |\widetilde{\mathtt{b}}|^2)\right)$. The matrix $A(\mathbf{u},\xi)$ can be symmetrized by

$$S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{1}{4\left(t+t_0^\hbar\right)^{n-2} f'} I_n \end{pmatrix},$$

which is symmetric and positive since $f' > 0$. We now reproduce the ideas of [8], inspired by hyperbolic theory, see e.g. [11]. For an integer $s > 2 + n/2$, we bound $(S\partial_x^\alpha \mathbf{u}^\hbar, \partial_x^\alpha \mathbf{u}^\hbar)$ where α is a multi index of length $\le s$, and $(\cdot,\cdot)$ is the usual L^2 scalar product. We have

$$\frac{d}{dt}\left(S\partial_x^\alpha \mathbf{u}^\hbar, \partial_x^\alpha \mathbf{u}^\hbar\right) = \left(\partial_t S\partial_x^\alpha \mathbf{u}^\hbar, \partial_x^\alpha \mathbf{u}^\hbar\right) + 2\left(S\partial_t\partial_x^\alpha \mathbf{u}^\hbar, \partial_x^\alpha \mathbf{u}^\hbar\right)$$

since S is symmetric. For the first term, we must consider the lower $n \times n$ block in S. Differentiating $(t+t_0^\hbar)^{2-n}$ yields a non-positive term, and we get

$$\left(\partial_t S\partial_x^\alpha \mathbf{u}^\hbar, \partial_x^\alpha \mathbf{u}^\hbar\right) \le \left\| \frac{1}{f'}\partial_t\left(f'\left(\left(t+t_0^\hbar\right)^n (|\widetilde{\mathtt{a}}^\hbar|^2 + |\widetilde{\mathtt{b}}^\hbar|^2)\right)\right)\right\|_{L^\infty} \left(S\partial_x^\alpha \mathbf{u}^\hbar, \partial_x^\alpha \mathbf{u}^\hbar\right) .$$

So long as $\|\mathbf{u}^\hbar\|_{L^\infty} \le 2\|a_0\|_{L^\infty}$, we have, for $t \le 2$ (to fix the ideas),

$$f'\left(\left(t+t_0^\hbar\right)^n (|\widetilde{\mathtt{a}}^\hbar|^2 + |\widetilde{\mathtt{b}}^\hbar|^2)\right) \ge \inf\left\{f'(y) \ ; \ 0 \le y \le 2^{n+2}\|a_0\|_{L^\infty}^2\right\} = \delta_n > 0\,,$$

where δ_n is now fixed, since f' is continuous with $f' > 0$. We infer, for $t \le 2$,

$$\left\| \frac{1}{f'}\partial_t\left(f'\left(\left(t+t_0^\hbar\right)^n (|\widetilde{\mathtt{a}}^\hbar|^2 + |\widetilde{\mathtt{b}}^\hbar|^2)\right)\right)\right\|_{L^\infty} \le C\left(\|\mathbf{u}^\hbar\|_{L^\infty}\right) \|\partial_t \mathbf{u}^\hbar\|_{L^\infty} \lesssim \|\mathbf{u}^\hbar\|_{H^s}\,,$$

where we used Sobolev embeddings and (4.7). For the second term we use

$$\left(S\partial_t\partial_x^\alpha \mathbf{u}^\hbar, \partial_x^\alpha \mathbf{u}^\hbar\right) = \frac{\hbar}{2}\left(SL(\partial_x^\alpha \mathbf{u}^\hbar), \partial_x^\alpha \mathbf{u}^\hbar\right) - \Big(S\partial_x^\alpha\Big(\sum_{j=1}^n A_j(\mathbf{u}^\hbar)\partial_j \mathbf{u}^\hbar\Big), \partial_x^\alpha \mathbf{u}^\hbar\Big).$$

We notice that SL is a skew-symmetric second order operator, so the first term is zero. The second term can be rewritten under the form

$$\Big(S\partial_x^\alpha\Big(\sum_{j=1}^n A_j(\mathbf{u}^\hbar)\partial_j\mathbf{u}^\hbar\Big),\partial_x^\alpha\mathbf{u}^\hbar\Big) = \Big(S\sum_{j=1}^n A_j(\mathbf{u}^\hbar)\partial_j\partial_x^\alpha\mathbf{u}^\hbar,\partial_x^\alpha\mathbf{u}^\hbar\Big)$$
$$+\Big(S\Big(\partial_x^\alpha(\sum_{j=1}^n A_j(\mathbf{u}^\hbar)\partial_j\mathbf{u}^\hbar) - \sum_{j=1}^n A_j(\mathbf{u}^\hbar)\partial_j\partial_x^\alpha\mathbf{u}^\hbar\Big),\partial_x^\alpha\mathbf{u}^\hbar\Big).$$

By symmetry of $SA_j(\mathbf{u}^\hbar)$,

$$\Big(S\sum_{j=1}^n A_j(\mathbf{u}^\hbar)\partial_j\partial_x^\alpha\mathbf{u}^\hbar,\partial_x^\alpha\mathbf{u}^\hbar\Big) = -\sum_{j=1}^n\Big(\partial_j(SA_j(\mathbf{u}^\hbar))\partial_x^\alpha\mathbf{u}^\hbar,\partial_x^\alpha\mathbf{u}^\hbar\Big)$$
$$-\sum_{j=1}^n\Big(SA_j(\mathbf{u}^\hbar)\partial_j\partial_x^\alpha\mathbf{u}^\hbar,\partial_x^\alpha\mathbf{u}^\hbar\Big).$$

Therefore, so long as $\|\mathbf{u}^\hbar\|_{L^\infty}\le 2\|a_0\|_{L^\infty}$ for $t\le 2$,

$$\left|\Big(S\sum_{j=1}^n A_j(\mathbf{u}^\hbar)\partial_j\partial_x^\alpha\mathbf{u}^\hbar,\partial_x^\alpha\mathbf{u}^\hbar\Big)\right| \lesssim \left\|\partial_x^\alpha\mathbf{u}^\hbar\right\|_{L^2}^2\left\|\nabla_x\mathbf{u}^\hbar\right\|_{L^\infty}\lesssim\left\|\mathbf{u}^\hbar\right\|_{H^s}^3 .$$

The usual estimates on commutators (see e.g. [11]) lead to

$$\left|\Big(S\Big(\partial_x^\alpha\Big(\sum_{j=1}^n A_j(\mathbf{u}^\hbar)\partial_j\mathbf{u}^\hbar\Big) - \sum_{j=1}^n A_j(\mathbf{u}^\hbar)\partial_j\partial_x^\alpha\mathbf{u}^\hbar\Big),\partial_x^\alpha\mathbf{u}^\hbar\Big)\right| \le C\left(\left\|\mathbf{u}^\hbar\right\|_{H^s}\right)\left\|\mathbf{u}^\hbar\right\|_{H^s}^2 .$$

Notice that S^{-1} can be bounded by $C(\|\mathbf{u}\|_{H^s})$, thus we have proved:

$$\frac{d}{dt}\sum_{|\alpha|\le s}\left(S\partial_x^\alpha\mathbf{u}^\hbar,\partial_x^\alpha\mathbf{u}^\hbar\right)\le C\left(\left\|\mathbf{u}^\hbar\right\|_{H^s}\right)\sum_{|\alpha|\le s}\left(S\partial_x^\alpha\mathbf{u}^\hbar,\partial_x^\alpha\mathbf{u}^\hbar\right) ,$$

for $s>2+d/2$. Gronwall lemma along with a continuity argument yield the counterpart of [8, Theorem 1.1]:

Proposition 4.2. *Under Assumptions 1.1 with $n>k>0$, let $s>2+n/2$. Then there exist $T>0$ independent of $\hbar\in]0,1]$ and $\psi^\hbar(t,x)=a^\hbar(t,x)e^{i\phi^\hbar(t,x)/\hbar}$ solution to (4.3) on $[t_0^\hbar,T+t_0^\hbar]$. Moreover, $a^\hbar$ and $\phi^\hbar$ are bounded in $L^\infty([t_0^\hbar,T+t_0^\hbar];H^s)$, uniformly in $\hbar\in]0,1]$.*

4.3. **Convergence and small time properties.** We can now complete the proof of Proposition 4.1. For $s>2+n/2$, we know that $\widetilde{a}^\hbar$ and $\widetilde{\mathtt{v}}^\hbar$ are bounded in $L^\infty([0,T];H^s)$, uniformly in $\hbar\in]0,1]$. Using (4.7), we infer that $\partial_t\widetilde{a}^\hbar$ and $\partial_t\widetilde{\mathtt{v}}^\hbar$ are bounded in $L^\infty([0,T];H^{s-2})$. Therefore, a subsequence of $(\widetilde{a}^\hbar,\widetilde{\mathtt{v}}^\hbar)$ converges uniformly in $C([0,T];H^{s'}_{\rm loc})$ to $(a,\mathtt{v})$ solution of (4.4) for any $s'<s-2$ (decreasing T if necessary, depending on the lifespan associated to (4.4), see e.g. [11, 12]). By uniqueness for (4.4), the whole sequence $(\widetilde{a}^\hbar,\widetilde{\mathtt{v}}^\hbar)$ is convergent. For $t_0^\hbar\le t\le T$, write

$$\mathtt{v}^\hbar(t,x)=\widetilde{\mathtt{v}}^\hbar(t,x)-\int_{t-t_0^\hbar}^t\partial_t\widetilde{\mathtt{v}}^\hbar(s,x)ds\ ;\ a^\hbar(t,x)=\widetilde{a}^\hbar(t,x)-\int_{t-t_0^\hbar}^t\partial_t\widetilde{a}^\hbar(s,x)ds$$

Still using the boundedness of $\partial_t\widetilde{a}^\hbar$ and $\partial_t\widetilde{\mathtt{v}}^\hbar$ in $L^\infty([0,T];H^{s-2})$, we deduce that the sequence $(a^\hbar,\mathtt{v}^\hbar)$ also converges to (4.4).

So far, we have not used the assumption $k>1$. It appears when one wants to approximate $e^{i\phi^\hbar/\hbar}$ by $e^{i\phi/\hbar}$: the factor $1/\hbar$ requires some care. This is where the analysis of (4.4) for small times comes into play.

Proposition 4.3. *Let Assumptions 1.1 be satisfied, with $n > k > 1$. Then there exist sequences $(\phi_j)_{j\geq 1}$ and $(a_j)_{j\geq 1}$ in $H^\infty(\mathbb{R}^n)$, such that the solution of (4.4) satisfies*

$$\phi(t,x) \sim \sum_{j\geq 1} t^{jn-1}\phi_j(x)\ ,\ \textit{and}\quad a(t,x)\sim \sum_{j\geq 0} t^{jn}a_j(x) \quad \textit{as } t\to 0\,.$$

This result follows easily from the proof of Proposition 4.2 and Borel lemma (see e.g. [13]); the first terms are computed in (5.1)–(5.3). Note that the series for a starts with $j=0$: the notations are consistent. We deduce

$$\phi^\hbar(t_0^\hbar,x)-\phi(t_0^\hbar,x) = -\phi(t_0^\hbar,x) = \mathcal{O}\left(\left(t_0^\hbar\right)^{n-1}\right) = \mathcal{O}\left(\hbar^{\frac{\gamma(n-1)}{1-\gamma}}\right)\,,$$

and by stability, for $t_0^\hbar \leq \tau \leq T$,

$$\sup_{t_0^\hbar\leq t\leq \tau}\left\|\phi^\hbar(t)-\phi(t)\right\|_{H^s(\mathbb{R}^n)} = \mathcal{O}\left(\hbar^{\frac{\gamma(n-1)}{1-\gamma}}\right) + \hbar\mathcal{O}(\tau)\,.$$

The last factor is due to the source term $\hbar\Delta a^\hbar$ in the error estimate between (4.6) and (4.4). Recalling that $\gamma = k/n$, we then have, for $s > n/2$,

$$\begin{aligned}\left\|\psi^\hbar - a_0 e^{i\phi/\hbar}\right\|_{L^\infty([t_0^\hbar,\tau];H^s)} &\lesssim \left\|a^\hbar - a\right\|_{L^\infty([t_0^\hbar,T];H^s)} + \left\|a-a_0\right\|_{L^\infty([t_0^\hbar,\tau];H^s)}\\ &\quad + \frac{1}{\hbar}\left\|\phi^\hbar-\phi\right\|_{L^\infty([t_0^\hbar,\tau];H^s)}\\ &\leq o(1) + \mathcal{O}(\tau^n) + \mathcal{O}\left(\hbar^{\frac{k-1}{1-\gamma}}\right) + \mathcal{O}(\tau)\,.\end{aligned}$$

This completes the proof of Proposition 4.1, and Theorem 1.7 follows.

Remark 4.4 (Well-prepared data). If we had $t_0^\hbar = 0$, then the assumption $k>1$ could be weakened to $k>0$. Back to the transform (4.1), if we assume that

$$u^\varepsilon_{|t=0} = a_0(x)e^{-i\frac{|x|^2}{2\varepsilon}}\exp\left(i\varepsilon^{\gamma-1}\phi\left(\varepsilon^\gamma,x\right)\right)\,,$$

then the $\mathcal{O}\left(\hbar^{\frac{k-1}{1-\gamma}}\right)$ term in the above estimate disappears, and we can conclude as before, supposing only $n>k>0$ (but still $n\geq 2$). Recall that if $0<k\leq 1$, then nonlinear effects are relevant at leading order for any positive time (nonlinear propagation); they show up precisely in the phase ϕ.

5. Stability issues

The construction of Section 3 and the results of the previous paragraph do not agree. To see this, we come back to Proposition 4.3: in (4.4), we have

$$\mathcal{O}\left(t^{n-2}\right):\quad \phi_1(x) = \frac{1}{n-1}f'(0)|a_0(x)|^2\,, \tag{5.1}$$

$$\mathcal{O}\left(t^{n-1}\right):\quad a_1 + \nabla\phi_1\cdot\nabla a_0 + \frac{1}{2}a_0\Delta\phi_1 = 0\,, \tag{5.2}$$

$$\mathcal{O}\left(t^{2n-2}\right):\quad (2n-1)\phi_2 + \frac{1}{2}|\nabla\phi_1|^2 + 2\operatorname{Re}(\overline{a_0}a_1)f'(0) + \frac{f''(0)}{2}|a_0|^4 = 0\,. \tag{5.3}$$

The function ϕ_1 is the same as the one obtained by the approach of Section 3: the two approximate solutions are close to each other up to the first boundary layer, when the first phase shift appears. On the other hand, we see that to get ϕ_2, the modulation of the amplitude (a_1) must be taken into account; in (3.3), g_2 is computed without evaluating Δa_0, unlike ϕ_2. This means in particular that the two approximate solutions diverge when reaching the second boundary layer: the approach of Section 3 is only formal, and does not lead to a good approximation.

And yet, the source term in Proposition 3.1 is small: thus, the linearized semi-classical Schrödinger operator is not stable, in the semi-classical limit. We will see below that this instability is not due to a spectral instability, but to the fact that the approach followed to construct the formal approximation was too crude.

This phenomenon is due to the super-criticality of the problem. Indeed, for fixed ε, we deal with a nonlinear Schrödinger equation with repulsive nonlinearity ($f' > 0$), for which global well-posedness results are available (see Remark 1.2). When using the transform (4.1), notice that the parameter $\hbar$ in (4.3) goes to zero as $\varepsilon \to 0$ only when $n > k$, that is in the super-critical case (compare with Section 2.2).

To understand better the instability mechanism, let us go back to the comparison between the construction of Section 3 and the results of the previous paragraph. Letting $N \to +\infty$ in Proposition 3.1, we have an approximate solution of the form

$$\begin{aligned} v^\varepsilon(t,x) &= \frac{e^{i\frac{|x|^2}{2\varepsilon(t-1)}}}{(1-t)^{n/2}} a_0\left(\frac{x}{1-t}\right) \exp\left(i\frac{1-t}{\varepsilon} g\left(\frac{\varepsilon^\gamma}{1-t},\frac{x}{1-t}\right)\right) \\ &= \frac{e^{-i\frac{|\xi|^2}{2\hbar\tau}}}{(1-t)^{n/2}} a_0(\xi) \exp\left(\frac{i}{\hbar}\frac{g(\tau,\xi)}{\tau}\right)\Big|_{(\tau,\xi)=\left(\frac{\varepsilon^\gamma}{1-t},\frac{x}{1-t}\right)}. \end{aligned}$$

This formula and the transform (4.1) show that the approximation of Section 3 is too crude, since it ignores the coupling between phase and amplitude for (4.3). Proposition 4.3 and (4.4) show that to have a good approximation of the phase, the coupling between phase and amplitude must be taken into account at *every* order.

We can go one step further in the understanding of this apparent instability, by applying the transform (4.1) to the intermediary approximate solution v_N^ε. We show that the formal approximation stops being a good approximation between the first and the second boundary layer. Assume $\sigma = 1$ so that the homogeneous nonlinearity satisfies Assumptions 1.1. Like for the exact solution, write

$$v_N^\varepsilon(t,x) = \frac{1}{(1-t)^{n/2}} \psi_N^\varepsilon\left(\frac{\varepsilon^\gamma}{1-t},\frac{x}{1-t}\right) e^{i\frac{|x|^2}{2\varepsilon(t-1)}}.$$

Using the expression (3.5), we check that $\psi_N^\hbar$ solves

$$i\hbar\partial_t\psi_N^\hbar + \frac{\hbar^2}{2}\Delta\psi_N^\hbar = t^{n-2}|\psi_N^\hbar|^2\psi_N^\hbar + \theta_N^\hbar(t,x),$$

along with the initial condition $\psi_N^\hbar\big|_{t=\hbar^{\frac{\gamma}{1-\gamma}}} = a_0(x) + \mathcal{O}\left(\hbar^{(\alpha-1)(1-\gamma)}\right)$ in $H^s(\mathbb{R}^n)$ for any $s > 0$, where:

$$\theta_N^\hbar(t,x) = \left(t^{(N+1)n-2}K_0(x) + i\hbar K_1(t,x)\right)\psi_N^\hbar(t,x) + i\hbar K_2(t,x) + \hbar^2 K_3(t,x),$$

for some "nice" functions K_j. Now write $\psi_N^\hbar(t,x) = a_N^\hbar(t,x)e^{i\phi_N^\hbar(t,x)/\hbar}$. We have:

$$\partial_t \mathbf{v}^\hbar + \sum_{j=1}^n A_j(\mathbf{v}^\hbar)\partial_j\mathbf{v}^\hbar = \frac{\hbar}{2}L\mathbf{v}^\hbar + \mathtt{S}^\hbar(t,x), \text{ with } \mathbf{v}^\hbar(t,x) = \begin{pmatrix} \operatorname{Re} a_N^\hbar \\ \operatorname{Im} a_N^\hbar \\ \partial_1\phi_N^\hbar \\ \vdots \\ \partial_n\phi_N^\hbar \end{pmatrix},$$

(5.4)

$$\text{and}\quad \mathtt{S}^\hbar(t,x) = (t+t_0^\hbar)\begin{pmatrix} K_1 + \operatorname{Re}\left((K_2 - i\hbar K_3)e^{i\phi_N^\hbar/\hbar}\right) \\ K_1 + \operatorname{Im}\left((K_2 - i\hbar K_3)e^{i\phi_N^\hbar/\hbar}\right) \\ -(t+t_0^\hbar)^{(N+1)n-3}\partial_1 K_0 \\ \vdots \\ -(t+t_0^\hbar)^{(N+1)n-3}\partial_n K_0 \end{pmatrix},$$

where the matrices A_j are the same as in Section 4.2 and the functions in the definitions of $\mathbf{v}^\hbar$ and $\mathtt{S}^\hbar$ are evaluated at $(t+t_0^\hbar, x)$. We can proceed like in Section 4.2: the new term is the source $\mathtt{S}^\hbar$. Unlike for the exact solution, the oscillatory aspect of the problem has not disappeared: the first two components of $\mathtt{S}^\hbar$ contain a highly oscillatory factor. Therefore, we cannot expect $\hbar$ independent energy estimates here. To measure the effect of this oscillatory term, forget the shift in time, and take $t_0^\hbar = 0$. Then assuming that for small times, $\partial_x^\alpha \phi_N^\hbar(t,x) = \mathcal{O}(t^{n-1})$ for any multi-index a (like for the exact solution), the H^s norms of the first two components of $\mathtt{S}^\hbar$ are controlled by

$$\mathcal{O}\left(t + \frac{t^{1+s(n-1)}}{\hbar^s}\right).$$

A source of order $\mathcal{O}(t)$ is not a problem, since we eventually consider the limit $t \to 0$. On the other hand, let us examine the last term. Back to the initial variables, this yields a control by

$$\left(\frac{\varepsilon^\gamma}{1-t}\right)^{1+s(n-1)} \varepsilon^{-s(1-\gamma)} = \frac{\varepsilon^{\gamma+s\alpha-s}}{(1-t)^{1+s(n-1)}}.$$

This is small for $1-t \gg \varepsilon^\omega$, with

$$\omega = \frac{\gamma+s\alpha-s}{1+s(n-1)}.$$

We check that for $n > \alpha = k > 1$, we have

$$\beta = \frac{\alpha-1}{n-1} < \omega = \frac{\gamma+s(\alpha-1)}{1+s(n-1)} < \frac{2\alpha-1}{2n-1}, \text{ for any } s \geq 1.$$

The first inequality means that we can expect the formal approximation to be a good approximation of the exact solution beyond the first boundary layer (and indeed, it is close to the approximate solution of Section 4). The second one explains why the approximation ceases to be relevant before the second boundary layer.

A possible way to understand the above computation is that the choice of the variables is crucial: working with the "usual" unknown v^ε (as in Section 3) is not very efficient. On the other hand, with the variables introduced by E. Grenier for his generalized WKB methods, a precise and rigorous analysis is possible, *via* the transform (4.1). Thus, adding new variables helps the analysis: this goes in the same direction as the general theory of geometric optics, and the recent approach followed by C. Cheverry for a refinement of this principle [4, 5].

6. After the cascade of phase shifts

As announced in the introduction, our analysis stops for times of order $t = 1-\lambda\varepsilon^\gamma$. For $\lambda \to +\infty$, we have Theorem 1.7. For bounded λ, $\lambda \in [\frac{1}{T}, +\infty[$, the first part of Proposition 4.1 provides an asymptotic description. This shows in particular that the solution $\psi^\hbar$ is approximated in terms of nonlinear geometric optics: the eikonal equation contains the amplitude, therefore the geometry of propagation needs not be the same as before, which occurred along rays $\frac{x}{1-t} = \text{const}$. Note also that the transform (4.1) changes the space variable into a parameterization of the family of rays, when they are straight lines. This explains Figure 1. Moreover, between the initial time $t=0$ and $t = 1-\frac{\varepsilon^\gamma}{T}$, the order of magnitude of u^ε changes. Indeed, Proposition 4.1 shows that for $t \in [t_0^\hbar, T]$, $\psi^\hbar$ is of order $\mathcal{O}(1)$ in $L^\infty(\mathbb{R}^n)$ (take $s > n/2$ and use Sobolev embeddings). By (4.1), we infer that the amplitude of u^ε varies like $(1-t)^{-n/2}$, and changes from $\mathcal{O}(1)$ initially, to $\mathcal{O}(\varepsilon^{-k/2})$ for $1-t \approx \varepsilon^\gamma$. Such an amplification is similar to what happens in the linear case.

The semi-classical conformal transform (4.1) cannot be interesting for values of t too close to 1, since it becomes singular. It seems reasonable to introduce the (L^2 unitary) scaling transform,

$$u^\varepsilon(t,x) = \frac{1}{\varepsilon^{n\gamma/2}}\varphi^\varepsilon\left(\frac{t-1}{\varepsilon^\gamma},\frac{x}{\varepsilon^\gamma}\right) = \frac{1}{\varepsilon^{k/2}}\varphi^\varepsilon\left(\frac{t-1}{\varepsilon^\gamma},\frac{x}{\varepsilon^\gamma}\right). \tag{6.1}$$

With the same change of notation as for ψ in Section 4.1, we have

$$i\hbar\partial_t\varphi^\hbar + \frac{\hbar^2}{2}\Delta\varphi^\hbar = f\left(|\varphi^\hbar|^2\right)\varphi^\hbar. \tag{6.2}$$

We now have exactly the same equation as in [8]. On the other hand, let us examine the initial condition. Taking into account the data $u^\varepsilon(0,x) = a_0(x)$, we find

$$\varphi^\hbar\left(-\hbar^{\frac{\gamma}{\gamma-1}},x\right) = \hbar^{\frac{k}{2(1-\gamma)}}a_0\left(\hbar^{\frac{\gamma}{1-\gamma}}x\right),$$

that one may try to decouple to

$$\varphi^\hbar(t,x) \sim \frac{1}{|t|^{n/2}}a_0\left(\frac{x}{t}\right)e^{i\frac{|x|^2}{2\hbar t}} \quad \text{as } t\to-\infty \text{ and } \hbar\to 0\,.$$

Of course, the above limits $t\to-\infty$ and $\hbar\to 0$ do not commute. Denote $U^\hbar(t)$ the unitary group associated to the linear semi-classical Schrödinger equation ($f\equiv 0$ in the above equation). Then for fixed $\hbar>0$,

$$U^\hbar(t)\varphi_0(x) \underset{t\to-\infty}{\sim} \frac{1}{(2i\pi\hbar t)^{n/2}}\widehat{\varphi}_0\left(\frac{x}{\hbar t}\right)e^{i\frac{|x|^2}{2\hbar t}}.$$

Fix $\hbar$ in the above asymptotics. If f has moderate growth as in Remark 1.2 (cubic nonlinearity in space dimension two or three for instance), then there is scattering for (6.2) with $\hbar$ fixed, and

$$\varphi^\hbar(0,x) \underset{\hbar\to 0}{\sim} \frac{1}{\hbar^{n/2}}\Phi\left(\frac{x}{\hbar}\right),$$

for some concentrating profile Φ, where the powers of $\hbar$ stem from the asymptotics for the free operator $U^\hbar$. We saw in the introduction that at least when the nonlinearity is homogeneous, the conservation of energy rules out such a possibility, in the limit $\hbar\to 0$.

It is probably more interesting to try to match with the results of Proposition 4.1. Comparing (4.1) and (6.1), Proposition 4.2 yields, for $\frac{-1}{t_0^\hbar}\leq t\leq\frac{-1}{T+t_0^\hbar}$:

$$\begin{aligned}\varphi^\hbar(t,x) &= \left(\frac{-1}{t}\right)^{n/2}\psi^\hbar\left(\frac{-1}{t},\frac{-x}{t}\right)e^{i\frac{|x|^2}{2\hbar t}}\\ &= \left(\frac{-1}{t}\right)^{n/2}a^\hbar\left(\frac{-1}{t},\frac{-x}{t}\right)\exp\left(\frac{i}{\hbar}\left(\frac{|x|^2}{2t}+\phi^\hbar\left(\frac{-1}{t},\frac{-x}{t}\right)\right)\right)\\ &=: \mathbf{a}^\hbar(t,x)e^{i\Phi^\hbar(t,x)/\hbar}.\end{aligned}$$

Note however that the term $\frac{|x|^2}{2t}$ in the phase does not belong to any Sobolev space; we have to adapt the statement of Proposition 4.2 before claiming that we have $\hbar$ independent estimates. We can then try to use Grenier's ideas again to extend the lifespan of $\mathbf{a}^\hbar$ and $\Phi^\hbar$ to $[\frac{-1}{T+t_0^\hbar},\widetilde{T}]$, with suitable $\hbar$ independent estimates. We shall not pursue this point of view here.

We conclude this section by listing a series of questions that remain:

Do we have a WKB like description of u^ε for some time $t>1$? If yes, with one or several phases?

Is there a caustic for $\varphi^\hbar$? This is not even clear. Indeed, the initial problem (1.2) contains a data which causes focusing at one point in the linear case. However, we saw above that when reaching the boundary layer of size ε^γ, phase and amplitude of the solution become coupled in such a way that the geometry of the propagation is modified. If by any chance there is no caustic for $\varphi^\hbar$, then we might take $\widetilde{T}$ arbitrarily large, and hope to get a description of u^ε for any time.

Note also that when WKB asymptotics is valid for $\varphi^\hbar$, then the nonlinear term in the conservation of the energy reaches its maximal order of magnitude. This is how we found the parameter γ in the introduction for an homogeneous nonlinearity. In the case of Assumptions 1.1, things are similar. Denote

$$F(y) = \int_0^y f\left(\eta^2\right)\eta d\eta \quad (\eta \text{ is a real variable}).$$

Then the generalization of the conservation of energy in (1.3) is:

$$\frac{1}{2}\|\varepsilon\nabla_x u^\varepsilon(t)\|_{L^2}^2 + \varepsilon^{-k}\int_{x\in\mathbb{R}^n} F\left(\varepsilon^{k/2}|u^\varepsilon(t,x)|\right)dx = \text{const.} = \mathcal{O}(1) \underset{\varepsilon\to 0}{\sim} \frac{1}{2}\|xa_0\|_{L^2}^2 .$$

The nonlinear term in the above energy is exactly $\int F\left(\left|\varphi^\hbar\left(\frac{t-1}{\varepsilon^\gamma},x\right)\right|\right)dx$, and is of order $\mathcal{O}(1)$ for, say, $\frac{t-1}{\varepsilon^\gamma} \in [\frac{-2}{T}, \frac{-1}{T}]$, where WKB asymptotics for $\varphi^\hbar$ stems from Proposition 4.1. (For $\frac{t-1}{\varepsilon^\gamma} \to -\infty$, dispersive properties of $\varphi^\hbar$ make the nonlinear term small.) On the other hand, we saw that if there is a caustic for $\varphi^\hbar$, then it cannot be reduced to a (single) point.

Finally, the apparent instability discussed in Section 5 suggests that computing reliable numerical simulations to understand the asymptotic behavior of u^ε is a challenging problem. Understanding the behavior of $(a^\hbar, \phi^\hbar)$ and $(\mathbf{a}^\hbar, \Phi^\hbar)$, rather than working on u^ε directly, would certainly be more reasonable.

References

1. R. Carles, *Geometric optics with caustic crossing for some nonlinear Schrödinger equations*, Indiana Univ. Math. J. **49** (2000), no. 2, 475–551.
2. ———, *Geometric optics and long range scattering for one-dimensional nonlinear Schrödinger equations*, Comm. Math. Phys. **220** (2001), no. 1, 41–67.
3. T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University Courant Institute of Mathematical Sciences, New York, 2003.
4. C. Cheverry, *Propagation of oscillations in real vanishing viscosity limit*, Comm. Math. Phys. **247** (2004), no. 3, 655–695.
5. ———, *Cascade of phases in turbulent flows*, Bull. Soc. Math. France (2005), to appear. Preprint version: `arXiv:math.AP/0402408`.
6. J. J. Duistermaat, *Oscillatory integrals, Lagrange immersions and unfolding of singularities*, Comm. Pure Appl. Math. **27** (1974), 207–281. MR 53 #9306
7. P. Gérard, *Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire*, Séminaire sur les Équations aux Dérivées Partielles, 1992–1993, École Polytech., Palaiseau, 1993, pp. Exp. No. XIII, 13.
8. E. Grenier, *Semiclassical limit of the nonlinear Schrödinger equation in small time*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 523–530.
9. J. Hunter and J. Keller, *Caustics of nonlinear waves*, Wave motion **9** (1987), 429–443.
10. S. Jin, C. D. Levermore, and D. W. McLaughlin, *The semiclassical limit of the defocusing NLS hierarchy*, Comm. Pure Appl. Math. **52** (1999), no. 5, 613–654.
11. A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Applied Mathematical Sciences, vol. 53, Springer-Verlag, New York, 1984.
12. G. Métivier and S. Schochet, *The incompressible limit of the non-isentropic Euler equations*, Arch. Ration. Mech. Anal. **158** (2001), no. 1, 61–90.
13. J. Rauch and M. Keel, *Lectures on geometric optics*, Hyperbolic equations and frequency interactions (Park City, UT, 1995), Amer. Math. Soc., Providence, RI, 1999, pp. 383–466.

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