Global convergence of a kinetic model for chemotaxis to a perturbed Keller-Segal model

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Abstract

We consider a class of kinetic models for chemotaxis with two positive non-dimensional parameters coupled to a parabolic equation for the chemo-attractant. If both parameters are set equal zero, we have the classical Keller-Segel model for chemotaxis. We prove global existence of solutions of this two-parameters kinetic model and prove convergence of this model to models for chemotaxis with global existence when one of these two parameters is set equal zero. In one case, we find as a limit model a kinetic models for chemotaxis while in the other case we find a perturbed Keller-Segel model with global existence of solutions.

1 Introduction

Mathematical models for chemotaxis, the cell movement induced by chemical substances, were introduced for the first time by Patlak [15], and further developed by Keller and Segel in references [10, 11]. Their model, from now on called the Keller-Segel model, consists in two parabolic (some times one parabolic and one elliptic) partial differential equations for the cell density $\rho(x,t)\geq 0$ and chemo-attractant density $S(x,t)\geq 0$, for $(x,t)\in\mathbb{R}^n\times\mathbb{R}_+$, $n=2$ or 3:

$$\begin{aligned}
\partial_t\rho &= \nabla\cdot\left(D_0\nabla\rho-G_\mu(\rho,S)\nabla S\right)\ , && (1)\\
\partial_t S &= \Delta S+\rho\ , && (2)\\
S(x,0) &= S^{\mathrm{I}}(x)\geq 0\ , && (3)\\
\rho(x,0) &= \rho^{\mathrm{I}}(x)\geq 0\ . && (4)
\end{aligned}$$

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The non-negative function $G_\mu(\rho, S)$ is known as the cross-diffusion term. One say that the solution (ρ, S) of the system (1–4) presents *finite-time blow up* when there is a time $T > 0$ such that

$$\lim_{t\to T^-} \left(||\rho(\cdot,t)||_{L^\infty(\mathbb{R}^n)} + ||S(\cdot,t)||_{L^\infty(\mathbb{R}^n)} \right) = \infty \ .$$

Although the exhibition of precise conditions that lead to blow up or to global existence of solutions is considered an important mathematical problem (see [7] and references therein), its physical or biological meaning is seldom made clear. Following Velazquez [16], we say that when finite-time blow up occurs, the model is an approximation of more realistic models without blow up, depending on a positive small non-dimensional parameter, such that when this parameter converges to zero, solutions present singular behavior.

It is know since [5, 12] that the classical Keller-Segel model (i.e., when $G_\mu(\rho, S) = \chi\rho$, where χ is a positive constant) in two dimensions can present blow up depending on the initial conditions.

Two different perturbations of the classical Keller-Segel model prevent blow up: the first one is due to Velazquez [16] and consists in considering a small parameter $\mu > 0$ and a cross-diffusion term given by

$$G_\mu(\rho, S) = \frac{1}{\mu} Q(\mu\rho) \ , \tag{5}$$

with

$$\begin{aligned} Q(y) &\approx y - \kappa y^2 \ , \text{as } y \to 0 \ , &(6)\\ \lim_{y\to\infty} Q(y) &< \infty \ . &(7) \end{aligned}$$

Then, one can prove global existence of solutions for the system (1–7) for $\mu > 0$. In the limit $\mu \to 0$ we have the classical Keller-Segel model. In this work, we call this model the Velazquez' model. See [16].

A second possibility is to consider a kinetic description of chemotaxis. Kinetic models for chemotaxis were introduced by Wolfgang Alt in [1, 2] and further developed by Alt *et al* in [13]. Kinetic models depend on a non-dimensional parameter $\varepsilon > 0$ such that, under mean assumptions, when $\varepsilon \to 0$ solutions converge to the solution of some associated Keller-Segel model. This was shown formally by Othmer and Hillen [6, 14] and rigorously first by Chalub *et al* [3] and then generalised by Hwang *et al* [8, 9].

Kinetic models consists in a transport equation for the phase space density $f(x, v, t)$, the density of cells in position x with velocity v at time t, associated to a given turning kernel $T_\varepsilon[S, f](x, v, v', t)$, the turning rate from v' to v, in a point (x, t) with chemo-attractant concentration and phase space density given by S and f, respectively:

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla f_\varepsilon = -\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon[S_\varepsilon, f_\varepsilon](f_\varepsilon) \ , \tag{8}$$

$$\mathcal{T}_\varepsilon[S,f](f)(x,v,t) := \int_V (T_\varepsilon[S,f](x,v',v,t)f(x,v,t) - T_\varepsilon[S,f](x,v,v',t)f(x,v',t))\, dv' \tag{9}$$

The macroscopic density is given by

$$\rho_\varepsilon(x,t) = \int_V f_\varepsilon(x,v,t)dv \ , \tag{10}$$

the chemo-attractant concentration obeys Equation (2) (with the sub-index ε in both S and ρ) and initial conditions are given by

$$f_\varepsilon(x,v,0) = f^{\mathrm{I}}(x,v) \geq 0 \ , \tag{11}$$

and by Equation (4). V is the compact and spherically symmetric set of all possible velocities.

It is possible to explicitly show a kinetic model with global existence of solutions such that its drift-diffusion limit $\varepsilon \to 0$ is a Keller-Segel model with blow up. See [3].

Velazquez' idea goes far beyond proving global existence for solutions of certain classes of Keller-Segel models. Namely, from the global existence of solutions of models like (1–7), he derived a dynamics for point masses in the limit $\mu \to 0$ that can be understood as a possible extension of the classical Keller-Segel model after the blow up time. See [16, 17]. In principle the same can be done with the model introduced in [3], and it is unclear if these results can be compared. In this work we show a class of models that include as limit cases these two and the extension of Keller-Segel model beyond the blow up time can, in principle, be done by setting ε and μ to 0. The important question if this extension depends on the precise way the limit $\varepsilon, \mu \to 0$ is taken remains open, and the current article shall be understood as a preparation to face that problem. This can be better visualised in Figure 1.

This paper is organised as follows: in Section 2, we introduce the models and show formally their limits. Finally, in Section 3, we prove rigorously the existence of both limits. We also show global existence of solutions when $\varepsilon, \mu > 0$, such that if $\varepsilon > 0$, L^∞-norms of the solutions are bounded by a μ-independent function and vice-versa.

2 Model and Formal limits

We introduce the turning kernel given by

$$T_{\varepsilon,\mu}[S,f](x,v,v',t) = \Phi(S(x + \varepsilon\zeta_\mu(f(x,v,t))v,t) - S(x,t))F(v) \tag{12}$$

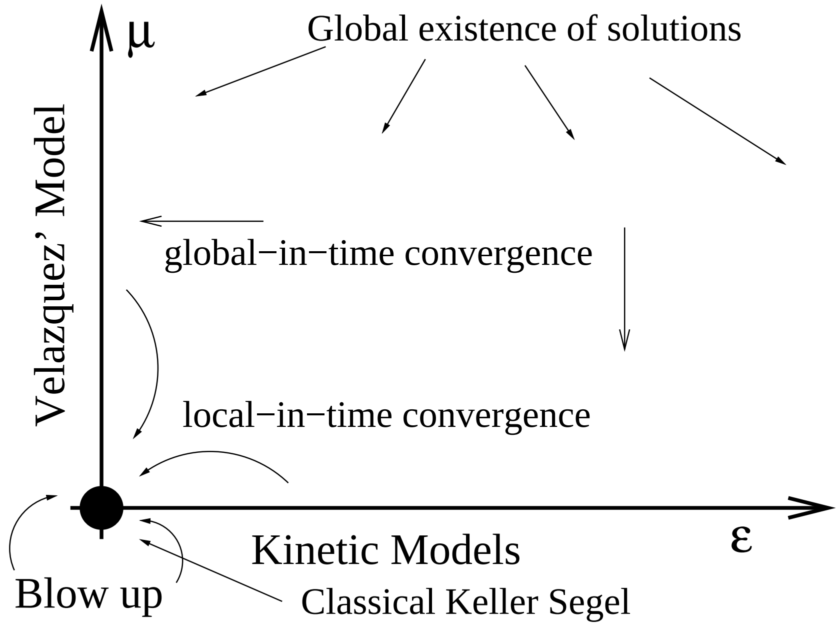

Figure 1: For any $\varepsilon, \mu > 0$, one can prove global existence of solution. These solutions converge to the solutions of the particular case when one of these two variables is set equal to zero. When $\mu, \varepsilon \to 0$ it is possible, in principle, to extend the classical Keller-Segel model beyond the blow up time. It is unclear if the extended model will depend or not on the way the limit is taken.

with

$$C_0(\mu) := \sup_{y \geq 0} \zeta_\mu(y) y \ , \quad \lim_{\mu \to 0} C_0(\mu) = \infty \ , \tag{13}$$

and for increasing function Φ such that $0 < \Phi_{\min} \leq \Phi(y) \leq Ay + B$, A and B positive constants and $\zeta_\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and bounded for all $\mu \geq 0$. We also assume that

$$\begin{aligned} F(v) &= F(|v|) > 0 \ , \\ \int_V F(v) dv &= 1 \ , \\ \int_V v F(v) dv &= 0 \ . \end{aligned}$$

Formally, we write $T_{\varepsilon,\mu}[S, f] = T^0_\mu[S, f] + \varepsilon T^1_\mu[S, f] + o(\varepsilon^2)$ such that

$$\begin{aligned} T^0_\mu[S, f] &= \Phi(0) F(v) \ , \\ T^1_\mu[S, f] &= (D\Phi)(0) \zeta_\mu(f) F(v) v \cdot \nabla S(x, t) \ . \end{aligned}$$

The formal proof is identical to the one presented in [3] and will not be repeated here. Only for completeness, we state that, at least formally, the drift-diffusion limit

$\varepsilon \to 0$ of the system (2–3), (6–12) and (14) is the system (1–7), with

$$D_0 = \frac{1}{\Psi(0)} \int_V v^2 F(v) dv \ ,$$
$$G_\mu(\rho, S) = \frac{(D\Phi)(0)\rho}{\Phi(0)} \int_V \zeta_\mu(\rho F(v)) F(v) v^2 dv \ .$$

From the last equation, we see that we should identify

$$\zeta_\mu(f) = \frac{\Phi(0)}{(D\Phi)(0)} \frac{1}{\int_V F(v) v^2 dv} \frac{Q(\mu f/F)}{\mu f/F} \ , \tag{14}$$

and the assumptions over ζ_μ are obviously satisfied.

Again formally, considering that

$$\zeta_0(y) := \lim_{\mu \to 0} \zeta_\mu(y) = \chi \ , \ \forall y \in \mathbb{R}_+ \ ,$$

where χ is a positive constant, we have as limit model, in the limit $\mu \to 0$ of the system (2–3), (8–13), the model (2–3), (8–11), with turning kernel given by

$$T_{\varepsilon,0}[S](x, v, v', t) = \Psi(S(x + \varepsilon v, t) - S(x, t)) F(v) \ ,$$

where we supposed, without lost of generality, that $\chi = 1$. This is exactly the model studied in references [3, 8].

3 Global-in-time Convergence to the Drift-diffusion Limit

Lemma 1. *Let $\rho \in L^p(\mathbb{R}^n \times [0, t])$ for any $t > 0$ and for p an even integer such that $2 \le p < \infty$. Then, for S solution of Equation (2) the following estimate holds:*

$$\sup_{s \in [0,t]} ||\nabla S(\cdot, s)||_{L^p(\mathbb{R}^n)} \le C ||\rho||_{L^p(\mathbb{R}^n \times [0,t])} \tag{15}$$

where $C = C(t, p)$.

Proof. After differentiating with respect to spatial variables x_k where $k = 1, 2, ..., n$, i.e. $\partial_t S_{x_k} - \Delta S_{x_k} = \rho_{x_k}$, by multiplying $S_{x_k}^{p-1}$ and using the integration by parts, we have

$$\frac{1}{p} \partial_t \int_{\mathbb{R}^n} S_{x_k}^p dx + \int_{\mathbb{R}^n} \nabla S_{x_k} \cdot \nabla \left(S_{x_k}^{p-1} \right) dx = - \int_{\mathbb{R}^n} \rho \left(S_{x_k}^{p-1} \right)_{x_k} dx$$

$$\frac{1}{p} \partial_t \int_{\mathbb{R}^n} S_{x_k}^p dx + (p-1) \int_{\mathbb{R}^n} |\nabla S_{x_k}|^2 S_{x_k}^{p-2} dx = -(p-1) \int_{\mathbb{R}^n} \rho S_{x_k, x_k} S_{x_k}^{p-2} dx$$

$$\leq (p-1)I \ ,$$

where, for simplicity, we denote $I = \int_{\mathbb{R}^n} \rho |S_{x_k,x_k}| S_{x_k}^{p-2} dx$. Then I can be estimated as follows:

$$\int_{\mathbb{R}^n} \rho |S_{x_k,x_k}| S_{x_k}^{p-2} dx \leq \left(\int_{\mathbb{R}^n} \rho^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |S_{x_x,x_k}|^2 S_{x_k}^{p-2} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} S_{x_k}^p dx \right)^{\frac{p-2}{2p}} \leq$$
$$\leq \frac{1}{p} \int_{\mathbb{R}^n} \rho^p dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla S_{x_x}|^2 |S_{x_k}|^{p-2} dx + \frac{p-2}{2p} \int_{\mathbb{R}^n} |S_{x_k}|^p dx$$

where we used that $a^x b^y c^z \leq xa + yb + zc$ in case $x+y+z=1$. Then, we obtain

$$\partial_t \int_{\mathbb{R}^n} S_{x_k}^p dx \leq C \left(\int_{\mathbb{R}^n} \rho^p + \frac{p-2}{2} \int_{\mathbb{R}^n} |\nabla S|^p dx \right) \ ,$$

where C is a constant depending on p. Applying the standard Gronwall's inequality, one can easily see that

$$\sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^n} \left(\partial_{x_k} S(x,\tau) \right)^p dx \leq e^{C(p-2)t/2} \int_0^t \int_{\mathbb{R}^n} \rho(x,\tau)^p dx d\tau \ .$$

Finally, we conclude that

$$\sup_{0 \leq \tau \leq t} ||\nabla S(\cdot,\tau)||_{L^p(\mathbb{R}^n)} \leq c_1 e^{c_2 t} ||\rho||_{L^p(\mathbb{R}^n \times [0,t])}.$$

□

Lemma 2. *For any fixed $\mu \geq 0$ and $\varepsilon > 0$ solutions of the kinetic model (2–3), (8–13) exist globally, i.e., for any $t > 0$, $f_{\varepsilon,\mu} \in L^\infty(0,t; L^\infty(\mathbb{R}^n \times V))$ and $S_{\varepsilon,\mu} \in L^\infty(0,t; L^\infty(\mathbb{R}^n))$. Furthermore, $||\rho_{\varepsilon,\mu}(\cdot,t)||_{L^\infty(\mathbb{R}^n)}$ and $||S_{\varepsilon,\mu}(\cdot,t)||_{L^\infty(\mathbb{R}^n)}$ are bounded by μ-independent functions.*

Proof. This can be done by following the same procedure as in [3] (see also [9]): Here, we only highlight the explicit ε dependence of $||f_{\varepsilon,\mu}(\cdot,\cdot,t)||_{L^\infty(\mathbb{R}^n \times V)}$. Namely, Equation (27) of [3] (or Equation (25) of [8]) is

$$||f_{\varepsilon,\mu}(\cdot,\cdot,t)||_{L^p(\mathbb{R}^n \times V)} \leq$$

$$||f^{\mathrm{I}}||_{L^p(\mathbb{R}^n \times V)} + \frac{C}{\varepsilon^2} \left(1 + \sup_{s \in [0,t]} ||S_{\varepsilon,\mu}(\cdot,t)||_{L^p(\mathbb{R}^n)} \right) \int_0^t ||f_{\varepsilon,\mu}(\cdot,\cdot,s)||_{L^p(\mathbb{R}^n \times V)} \, ds \ ,$$

for $p \in [2,\infty]$. We proceed as in the cited references and conclude, using Gronwall's inequality, the existence of a μ-independent global bound for $||f_{\varepsilon,\mu}(\cdot,\cdot,t)||_{L^\infty(\mathbb{R}^n \times V)}$. We use that $||\rho(\cdot,t)||_{L^\infty(\mathbb{R}^n)} \leq c||f(\cdot,\cdot,t)||_{L^\infty(\mathbb{R}^n \times V)}$ and the first bound follows. The existence of μ-independent global bound for $||S_{\varepsilon,\mu}(\cdot,t)||_{L^\infty(\mathbb{R}^n)}$ is consequence of Equation (2). □

Corollary 1. *In the limit $\mu \to 0$, solutions of the kinetic model (2–3), (8–13), converge in $L^\infty_{\mathrm{loc}}(0,t;L^\infty(\mathbb{R}^n))$ to solutions of the kinetic model (2–3), (8–11).*

Theorem 1. *For any $\mu > 0$, let $(f_{\varepsilon,\mu}, S_{\varepsilon,\mu})$ be the solution of the kinetic model (2–3), (8–13) and let $\rho_{\varepsilon,\mu}$ be given by Equation (10). Then*

$$\begin{aligned} \rho_{\varepsilon,\mu} &\to \rho_{0,\mu} \ \ \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^n \times (0,t)) \ , \\ S_{\varepsilon,\mu} &\to S_{0,\mu} \ \ \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^n \times (0,t)) \ , \\ \nabla S_{\varepsilon,\mu} &\to \nabla S_{0,\mu} \ \ \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^n \times (0,t)) \ , \end{aligned}$$

in the limit $\varepsilon \to 0$, where $(\rho_{0,\mu}, S_{0,\mu})$ is the solution of the Velazquez' model (1–7), with the same fixed $\mu > 0$, for any time interval $[0,t]$ with $0 < t < \infty$.

Proof. For simplicity, we use the following abbreviations:

$$\Phi := \Phi(S_{\varepsilon,\mu}(x+\varepsilon\zeta_\mu(f_{\varepsilon,\mu})v,t) - S_{\varepsilon,\mu}(x,t)), \quad \Phi' := \Phi(S_{\varepsilon,\mu}(x+\varepsilon\zeta_\mu(f'_\varepsilon)v',t) - S_{\varepsilon,\mu}(x,t)) \ ,$$

for $f_{\varepsilon,\mu} := f_{\varepsilon,\mu}(x,v,t)$ and $f'_{\varepsilon,\mu} := f_{\varepsilon,\mu}(x,v',t)$. The anti-symmetric and symmetric parts of the turning kernel (12) are given by

$$\phi^A_{\varepsilon,\mu} = (\Phi - \Phi')\,\frac{F(v)F(v')}{2}, \qquad \phi^S_{\varepsilon,\mu} = (\Phi + \Phi')\,\frac{F(v)F(v')}{2}.$$

We note that the symmetric part is bounded from below, i.e. $\phi^S_{\varepsilon,\mu} \ge \Phi_{\min}F(v)F(v')$, and we rewrite the anti-symmetric part as follows:

$$\phi^A_{\varepsilon,\mu} = (I_A - J_A)\frac{F(v)F(v')}{2}, \quad I_A = \Phi - \Phi(0), \ \ J_A = \Phi' - \Phi(0).$$

We consider first I_A.

$$I_A = \int_0^1 \frac{d}{dz}\Phi(S_{\varepsilon,\mu}(x+z\varepsilon\zeta_\mu(f_{\varepsilon,\mu})v,t) - S_{\varepsilon,\mu}(x,t))dz$$

$$= \int_0^1 (D\Phi)(S_{\varepsilon,\mu}(x+z\varepsilon\zeta_\mu(f_{\varepsilon,\mu})v,t) - S_{\varepsilon,\mu}(x,t))\nabla S_{\varepsilon,\mu}(x+z\varepsilon\zeta_\mu(f_{\varepsilon,\mu})v,t)dz\cdot v\varepsilon\zeta_\mu(f_{\varepsilon,\mu}).$$

In a similar manner, J_A can be written as

$$J_A = \int_0^1 \frac{d}{dz}\Phi(S_{\varepsilon,\mu}(x+z\varepsilon\zeta_\mu(f'_{\varepsilon,\mu})v',t) - S_{\varepsilon,\mu}(x,t))dz$$

$$= \int_0^1 (D\Phi)(S_{\varepsilon,\mu}(x+z\varepsilon\zeta_\mu(f'_{\varepsilon,\mu})v',t) - S_{\varepsilon,\mu}(x,t))\nabla S_{\varepsilon,\mu}(x+z\varepsilon\zeta_\mu(f'_{\varepsilon,\mu})v',t)dz\cdot v'\varepsilon\zeta_\mu(f'_{\varepsilon,\mu}).$$

Using $|(D\Phi)(\cdot)| \leq C$ and that v is compactly supported, we have

$$\left(\phi_\varepsilon^A\right)^2 \leq C\varepsilon^2\zeta_\mu^2(f_{\varepsilon,\mu})\left[\int_0^1 |\nabla S_{\varepsilon,\mu}(x+z\varepsilon\zeta_\mu(f_{\varepsilon,\mu})v,t)|^2 dz + \int_0^1 |\nabla S_{\varepsilon,\mu}(x+z\varepsilon\zeta_\mu(f'_{\varepsilon,\mu})v',t)|^2 dz\right]$$

As in Equation (28) in reference [3], we write

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^n}\int_V \frac{f_{\varepsilon,\mu}^p}{F^{p-1}} dvdx \leq \frac{C}{\varepsilon^2}\int_{\mathbb{R}^n}\int_V\int_V \frac{\left(\phi_{\varepsilon,\mu}^A\right)^2}{\phi_{\varepsilon,\mu}^S}\frac{f_{\varepsilon,\mu}^p}{F^p} dv'dvdx$$

$$\leq \frac{C}{\mu^2}\int_{\mathbb{R}^n}\int_V\int_V \frac{f_{\varepsilon,\mu}^{p-2}}{F^p}\left[\int_0^1 |\nabla S_{\varepsilon,\mu}(x+z\varepsilon\zeta_\mu(f_{\varepsilon,\mu})v,t)|^2 dz + \int_0^1 |\nabla S_{\varepsilon,\mu}(x+z\varepsilon\zeta_\mu(f'_{\varepsilon,\mu})v',t)|^2 dz\right] dv'dvdx$$

$$\leq \frac{C}{\mu^2}\int_0^1\int_V\int_V\int_{\mathbb{R}^n} \frac{f_{\varepsilon,\mu}^{p-2}}{F^p}\left[|\nabla S_{\varepsilon,\mu}(x+z\varepsilon\zeta_\mu(f_{\varepsilon,\mu})v,t)|^2 dz + |\nabla S_{\varepsilon,\mu}(x+z\varepsilon\zeta_\mu(f'_{\varepsilon,\mu})v',t)|^2 dz\right] dxdv'dvdz$$

where we changed the order of integration. Now we choose $q = p/2$ and $q' = p/(p-2)$ such that q and q' are Hölder exponents. Using Hölder inequality with these exponents, we obtain

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^n}\int_V \frac{f_{\varepsilon,\mu}^p}{F^{p-1}} dvdx \leq \frac{C}{\mu^2}\int_0^1\int_V\int_V \frac{1}{F^p}\left(\int_{\mathbb{R}^n} |\nabla S_{\varepsilon,\mu}(x,t)|^p dx\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^n} f_{\varepsilon,\mu}^p dx\right)^{\frac{q-2}{q}}$$

$$\leq \frac{C}{\mu^2}\left(\int_{\mathbb{R}^n} |\nabla S_{\varepsilon,\mu}(x,t)|^p dx\right)^{\frac{2}{p}}\left(\int_{\mathbb{R}^n}\int_V \frac{f_{\varepsilon,\mu}^p}{F^{p-1}} dvdx\right)^{\frac{p-2}{p}},$$

where we used that F is bounded from below and the order of integration is changed. We integrate in time and find that

$$\int_{\mathbb{R}^n}\int_V \frac{f_{\varepsilon,\mu}^p}{F^{p-1}} dvdx \leq \int_{\mathbb{R}^n}\int_V \frac{(f^{\mathrm{I}})^p}{F^{p-1}} dvdx +$$

$$\frac{C}{\mu^2}\sup_{s\in[0,t]}\left(||\nabla S_{\varepsilon,\mu}(\cdot,s)||_{L^p(\mathbb{R}^n)}^p\right)^{2/p}\left[\int_0^t\int_{\mathbb{R}^n}\int_V \frac{f_{\varepsilon,\mu}^p}{F^{p-1}} dvdxds\right]^{(p-2)/p}.$$

Finally, we use Lemma 1 and the fact that

$$||\rho(\cdot,t)||_{L^p(\mathbb{R}^n)} \leq C||f(\cdot,\cdot,t)||_{L^p(\mathbb{R}^n\times V)}$$

to conclude that there exists a constant C, depending on p, t such that

$$\int_{\mathbb{R}^n}\int_V \frac{f_{\varepsilon,\mu}^p}{F^{p-1}} dvdx \leq \int_{\mathbb{R}^n}\int_V \frac{(f^{\mathrm{I}})^p}{F^{p-1}} dvdx + \frac{C}{\mu^2}\int_0^t\int_{\mathbb{R}^n}\int_V \frac{f_{\varepsilon,\mu}^p}{F^{p-1}} dvdxds\ .$$

Now, we use the standard Gronwall's inequality and conclude that $f_{\varepsilon,\mu}/F \in L^p(\mathbb{R}^n; F dv dx)$, $\forall t \in \mathbb{R}_+$, for even $p \in [2, \infty)$.

To conclude the proof, we only need the strong convergence of $f_{\varepsilon,\mu}$ to $f_{0,\mu}$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. But this is a simple consequence of the strong convergence of $\rho_{\varepsilon,\mu}$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ to $\rho_{0,\mu}$ (see [4]) and the boundedness of the remainder term

$$r_{\varepsilon,\mu} := \frac{f_{\varepsilon,\mu} - \rho_{\varepsilon,\mu} F}{\varepsilon}$$

(see [3]). Then, using that $f_{0,\mu} = \rho_{0,\mu} F$, we write

$$f_{\varepsilon,\mu} - f_{0,\mu} = \varepsilon r_\varepsilon + (\rho_{\varepsilon,\mu} - \rho_{0,\mu}) F \ ,$$

and conclude that $f_{\varepsilon,\mu} \to f_{0,\mu}$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$.

$\square$

Remark 1. *The main difference between the result presented in Theorem 1 and the ones in References [3] and [8] is that here we prove the global in time convergence from the Kinetic model to the Velazquez' models, while before it was possible to prove only the local-in-time convergence, because the limit model (i.e., the classical Keller-Segel model) presents finite-time blow up.*

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