Special Solutions for Linear Functional Differential Equations and Asymptotic Behaviour

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Abstract

The present paper deals with special solutions for linear functional differential equations with small delays. The importance of special solutions relies on the fact that, under some conditions, they can be used to describe the asymptotic behaviour of all solutions. Sufficient conditions for the existence of special solutions for non-autonomous equations in $\mathbb{R}^n$ are given. For autonomous equations in $\mathbb{R}^n$, results relating special solutions and characteristic values are presented. We also consider linear autonomous functional differential equations in Banach spaces, and use special solutions to study the asymptotic behaviour of their solutions. Several applications are given.

1. Introduction

Let $C = C([-r,0];\mathbb{R}^n)$ be the space of continuous functions from $[-r,0]$ to $\mathbb{R}^n$, $r > 0$, equipped with the sup norm $\|\phi\| = \max_{-r\le\theta\le 0}|\phi(\theta)|$. Consider a general non-autonomous linear functional differential equation (FDE) in $\mathbb{R}^n$,

$$\dot{x}(t) = \tilde{L}(t)x_t, \tag{1.1}$$

where $\mathbb{R} \times C \ni (t,\phi) \mapsto \tilde{L}(t)\phi \in \mathbb{R}^n$ is continuous, and $\tilde{L}(t)$ a linear operator for all $t \in \mathbb{R}$. As usual, $x_t \in C$ is defined by $x_t(\theta) = x(t+\theta), -r \le \theta \le 0$. A solution $u(t)$ of (1.1) is said to be a *special solution* if $u(t)$ is a solution defined on $\mathbb{R}$ satisfying $\sup_{t\le 0}|u(t)|e^{t/r} < \infty$. Several authors

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investigated the properties of special solutions of FDEs in $\mathbb{R}^n$ with small delays (see e.g. [1, 2, 3, 6, 7, 12]), since they can be used to characterize the asymptotic behaviour of all solutions. In fact, under the assumption

$$\|\tilde{L}(t)\| \le l \quad \text{for almost every } (a.e.)\ t \in \mathbb{R} \quad \text{and} \quad ler < 1\,, \tag{1.2}$$

Driver (see [3] and references therein) proved the existence of exactly n linearly independent special solutions for (1.1) (up to a shift in time), and that all other solutions approach, in an exponential way, some special solution as $t \to \infty$. To be more precise, there is a unique $n \times n$ matrix-valued function $X(t)$, called the (Ryabov) special matrix solution, whose columns are linearly independent special solutions and such that $X(0)$ is the identity matrix. Moreover, solutions $x(t) = x(\phi)(t)$ of (1.1) with initial conditions $\phi \in C$ at zero satisfy $X^{-1}(t)x(t) \to \ell(\phi)$ as $t \to \infty$, for some vector $\ell(\phi) \in \mathbb{R}^n$ [3]. It was also shown in [1] that the matrix $X(t)$ is the fundamental matrix solution of a system of n linear homogeneous ordinary differential equations. In [2], Arino and Pituk were able to give the limit $\ell(\phi)$ explicitly in terms of the initial condition ϕ, therefore to get an explicit asymptotic representation of all solutions of (1.1).

The works of Driver [3] and Arino and Pituk [2] were a big motivation for the present paper. Here, we establish some improvements in their results on special solutions of linear FDEs in $\mathbb{R}^n$ with small delays, and further analyze the autonomous situation. Another purpose of this paper is to study special solutions for linear FDEs in Banach spaces, and use them to address the asymptotic behaviour of all solutions.

A brief summary of the paper is as follows. In Section 2, we restrict our attention to linear FDEs (1.1) in $\mathbb{R}^n$, and suppose that (possibly infinite countable) time-dependent discrete delays as well as distributed delays are incorporated in $\tilde{L}$. Note that FDEs with both discrete and distributed delays appear in many applications, such as neural network and population dynamics models. By separating the discrete from the distributed delays, the results in [2, 3], referring to the number of linearly independent special solutions and the asymptotic behaviour of solutions of (1.1), are recovered under a hypothesis on the size of the delays slightly weaker than (1.2). In Section 2, some proofs are omitted, since they are a straightforward generalization of the prooofs in [2, 3].

It turns out from [2] that the role of special solutions of (1.1) is similar to the role played by solutions of exponential form for autonomous linear FDEs in $\mathbb{R}^n$. In fact, for an equation $\dot{x}(t) = \tilde{L}x_t$, the well-known formal adjoint theory [9] is a powerful tool for studying the asymptotic behaviour of all solutions, and clearly there is a relationship between its special solutions and its characteristic values. This situation will be exploited in more detail in Section 3. If the delay r

is small, we shall show that the space of special solutions is the generalized eigenspace associated with the set Λ of the characteristic values λ with $Re\,\lambda \geq -1/r$.

In Section 4, we deal with autonomous linear FDEs in Banach spaces. Note that FDEs in Banach spaces are particularly important in applications, since they include reaction-diffusion equations with delays. For $\mathcal{X}$ a Banach space, take $C = C([-r,0];\mathcal{X})$ with the sup norm as the phase space , and consider autonomous FDEs in C written in abstract form as

$$\dot{u}(t) = A_T u(t) + \tilde{L} u_t, \tag{1.3}$$

where A_T is the infinitesimal generator of a C_0-semigroup of linear operators on $\mathcal{X}$, and $\tilde{L}: C \to \mathcal{X}$ is linear bounded. For (1.3), again we separate discrete from distributed delays in $\tilde{L}$, and assume hypotheses similar to the ones in [14] (see hypotheses (H1)-(H4) in Section 4). Some additional conditions on the eigenvalues of A_T are imposed, in order to deduce the asymptotic behaviour of all solutions of (1.3).

For both FDEs in $\mathbb{R}^n$ and in Banach spaces, the results presented are illustrated with applications.

Throughout the paper, given Banach spaces X, Y, we denote by $\mathcal{L}(X,Y)$ the Banach space of linear bounded operators on X into Y, with the operator norm. The space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ of the $n \times n$ constant matrices is simply denoted by Mat_n.

2. Special solutions of linear FDEs in $\mathbb{R}^n$ and asymptotic behaviour

Let $C := C([-r,0];\mathbb{R}^n)$ be the space of continuous functions from $[-r,0]$ to $\mathbb{R}^n$, $r > 0$, equipped with the sup norm $\|\phi\| = \max_{-r\leq\theta\leq 0} |\phi(\theta)|$. Consider a general non-autonomous linear FDE (1.1), where $\tilde{L}: \mathbb{R} \to \mathcal{L}(C, \mathbb{R}^n)$ is continuous. We suppose that (possibly infinite countable) time-dependent discrete delays as well as distributed delays are incorporated in $\tilde{L}$; separating the discrete from the distributed delays, we write

$$\tilde{L}(t)\phi = \sum_{i=1}^{\infty} M_i(t)\phi(-\tau_i(t)) + L(t)\phi, \quad t \in \mathbb{R},\, \phi \in C,$$

so (1.1) is written in the form

$$\dot{x}(t) = \sum_{i=1}^{\infty} M_i(t)x(t-\tau_i(t)) + L(t)x_t, \tag{2.1}$$

where: $\tau_i : \mathbb{R} \to \mathbb{R}$, $M_i : \mathbb{R} \to Mat_n$ $(i \in \mathbb{N})$ and $L : \mathbb{R} \to \mathcal{L}(C, \mathbb{R}^n)$ are continuous functions; $\sum_{i=1}^{\infty} \|M_i(t)\| < \infty$, and $0 \leq \tau_i(t) \leq r_i \leq r$, for all $t \in \mathbb{R}$. For (2.1), an initial condition at $t = t_0$ has the form $x_{t_0} = \phi$ where $\phi \in C$, i.e., $x(t_0 + \theta) = \phi(\theta)$ for $\theta \in [-r, 0]$.

For Eq. (2.1), the following hypotheses are assumed:

there are constants $m_i \geq 0,\ i \in \mathbb{N}$, and $l \geq 0$ (with $\sum_{i=1}^{\infty} m_i + l > 0$), such that

$$\|M_i(t)\| \leq m_i, \quad \|L(t)\| \leq l, \quad \text{for a.e. } t \in \mathbb{R} \tag{2.2}$$

and

$$\sum_{i=1}^{\infty} m_i e^{r_i/r} + le < \frac{1}{r}. \tag{2.3}$$

Definition 2.1. A solution $u(t)$ of (2.1) is called a **special solution** if $u(t)$ is defined on $\mathbb{R}$ and $|u(t)|e^{t/r}$ is bounded for $t \leq 0$.

Note that both cases $\dot{x}(t) = L(t)x_t$ and $\dot{x}(t) = \sum_{i=1}^{\infty} M_i(t)x(t - \tau_i(t))$ are included in (2.1). For results on special solutions for equations of the form $\dot{x}(t) = L(t)x_t$, see [1–3]. In the following, results similar to the ones in [3] and [2], referring to the number of linearly independent special solutions and to the asymptotic behaviour of solutions, are recovered for Eq. (2.1) under hypotheses (2.2) and (2.3), rather than the more restrictive one $(\sum_{i=1}^{\infty} m_i + l)er < 1$ in [3] (cf. (1.2)). Some proofs are omitted, since they follow by adjusting the ideas in [2, 3] to the present situation. However, some proofs are included, for the sake of completeness as well as due to the generalizations in Section 4.

Theorem 2.1. *Assume (2.2)-(2.3). For each $t_0 \in \mathbb{R}, b \in \mathbb{R}^n$, there is a unique special solution $u(t)$ of (2.1) such that $u(t_0) = b$.*

Proof. We adapt the proof of [3, Theorem 1], in order to use a contraction principle argument. Fix $t_0 \in \mathbb{R}, b \in \mathbb{R}^n$. Consider the Banach space

$$\mathcal{S} = \{u \in C((-\infty, t_0]; \mathbb{R}^n]) : |u(t)|e^{t/r} \text{ is bounded}\},$$

with the norm

$$\|u\|_{\mathcal{S}} = \sup_{t \leq t_0} |u(t)|e^{t/r},$$

and the operator $\mathcal{T}_{(t_0,b)}$, denoted below simply by $\mathcal{T}$, defined by

$$\mathcal{T}u(t) = b + \int_{t_0}^{t} \Big(\sum_{i=1}^{\infty} M_i(s)u(s - \tau_i(s)) + L(s)u_s \Big) ds, \quad t \leq t_0, u \in \mathcal{S}.$$

For $t \leq t_0, u \in \mathcal{S}$, we have $|u(t - \tau_i(t))| \leq e^{(r_i - t)/r}\|u\|_{\mathcal{S}}$ and $\|u_t\| \leq e^{1-t/r}\|u\|_{\mathcal{S}}$. Thus,

$$\begin{aligned} |\mathcal{T}u(t)| &\leq |b| + \int_{t}^{t_0} \Big(\sum_{i=1}^{\infty} m_i |u(s - \tau_i(s))| + l\|u_s\| \Big) ds \\ &\leq |b| + \Big(\sum_{i=1}^{\infty} m_i e^{r_i/r} + le \Big) \|u\|_{\mathcal{S}} \int_{t}^{t_0} e^{-s/r} ds, \quad t \leq t_0, u \in \mathcal{S}, \end{aligned}$$

which implies that $\mathcal{T}u \in \mathcal{S}$, with

$$\|\mathcal{T}u\|_{\mathcal{S}} \le |b|e^{t_0/r} + \Big(\sum_{i=1}^{\infty} m_i e^{r_i/r} + le\Big) r\|u\|_{\mathcal{S}}.$$

We now prove that $\mathcal{T}$ is a contraction on $\mathcal{S}$. For $u, v \in \mathcal{S}, t \le t_0$, in a similar way we deduce that

$$|\mathcal{T}u(t) - \mathcal{T}v(t)| \le \Big(\sum_{i=1}^{\infty} m_i e^{r_i/r} + le\Big)\|u - v\|_{\mathcal{S}} \int_t^{t_0} e^{-s/r} ds,$$

hence

$$\|\mathcal{T}u - \mathcal{T}v\|_{\mathcal{S}} \le \Big(\sum_{i=1}^{\infty} m_i e^{r_i/r} + le\Big) r\|u - v\|_{\mathcal{S}},$$

and from (2.3) it follows that $\mathcal{T}$ is a contraction. Clearly, the unique fixed point u of $\mathcal{T}$ is a solution of (2.1) defined on $(-\infty, t_0]$, with $u(t_0) = b$ and $|u(t)|e^{t/r}$ bounded for $t \le t_0$. On the other hand, there is a unique solution $x(t)$ of (2.1) on $[t_0, \infty)$ satisfying the initial condition $x_{t_0} = u_{t_0}$. This means that $u(t)$ can be extended uniquely as a solution of (2.1) on $\mathbb{R}$. ■

The next lemma plays an important role for the characterization of special solutions, and hence also for the asymptotic behaviour of all solutions of (2.1) as $t \to \infty$.

Lemma 2.2. *Assume (2.2)-(2.3), and for $t_0 \in \mathbb{R}, b \in \mathbb{R}^n$ denote by $u(\cdot, t_0, b)$ the special solution of (2.1) such that $u(t_0, t_0, b) = b$. Then, for any $t_0 \in \mathbb{R}, b \in \mathbb{R}^n$,*

$$|u(t, t_0, b)| \le |b|e^{\lambda_0(t - t_0)}, \qquad t \le t_0, \tag{2.4}$$

where $\lambda_0 \in (-1/r, 0)$ is defined by

$$\sum_{i=1}^{\infty} m_i e^{-\lambda_0 r_i} + le^{-\lambda_0 r} = -\lambda_0. \tag{2.5}$$

Proof. We first note that $g(\lambda) := \sum_{i=1}^{\infty} m_i e^{-\lambda r_i} + le^{-\lambda r} + \lambda$ is well defined on $[-1/r, 0]$, with $g(0) > 0, g(-1/r) = \sum_{i=1}^{\infty} m_i e^{r_i/r} + le - 1/r < 0$ and $g'(\lambda) > -r(\sum_{i=1}^{\infty} m_i e^{r_i/r} + le) + 1 > 0$ for $\lambda \in (-1/r, 0)$, thus there is a unique $\lambda_0 \in (-1/r, 0)$ satisfying $g(\lambda_0) = 0$.

Consider the notations in the above proof. By the contraction principle, the sequence of functions $(u^{(n)})$ defined by

$$u^{(0)}(t) = b, \quad u^{(n+1)}(t) = (\mathcal{T}_{(t_0,b)} u^{(n)})(t),\ n \in \mathbb{N}_0, \qquad \text{for } t \le t_0,$$

converges on $\mathcal{S}$ to $u(\cdot, t_0, b)$. By induction, we shall prove that, for $n \in \mathbb{N}_0$,

$$|u^{(n)}(t)| \le |b|e^{\lambda_0(t - t_0)}, \qquad t \le t_0. \tag{2.6}$$

Clearly, (2.6) is true for $n = 0$. Assuming it holds for $n \in \mathbb{N}_0$, we get

$$
\begin{aligned}
|u^{(n+1)}(t)| &\le |b| + \int_t^{t_0} \Big[\sum_{i=1}^{\infty} \Big| M_i(s) u^{(n)}(s-\tau_i(s)) \Big| + |L(s) u_s^{(n)}| \Big] ds \\
&\le |b| + \int_t^{t_0} \Big[\sum_{i=1}^{\infty} m_i |u^{(n)}(s-\tau_i(s))| + l \sup_{\theta \in [-r,0]} |u^{(n)}(s+\theta)| \Big] ds \\
&\le |b| \Big[1 + e^{-\lambda_0 t_0} \big(\sum_{i=1}^{\infty} m_i e^{-\lambda_0 r_i} + l e^{-\lambda_0 r} \big) \int_t^{t_0} e^{\lambda_0 s} ds \Big] \\
&= |b| \Big[1 + \frac{1}{\lambda_0} \big(\sum_{i=1}^{\infty} m_i e^{-\lambda_0 r_i} + l e^{-\lambda_0 r} \big) (1 - e^{\lambda_0 (t-t_0)}) \Big], \qquad t \le t_0.
\end{aligned}
$$

Since λ_0 satisfies (2.5), we obtain

$$
|u^{(n+1)}(t)| \le |b| e^{\lambda_0 (t-t_0)}, \qquad t \le t_0,
$$

so (2.6) is proven for $n \in \mathbb{N}_0$. By letting $n \to \infty$, from (2.6) we derive (2.4). ■

For $t_0 \in \mathbb{R}, b \in \mathbb{R}^n$, as above let $u(t, t_0, b)$, $t \in \mathbb{R}$, be the special solution of (2.1) with $u(t_0, t_0, b) = b$. From Theorem 2.1, it follows that

$$
u(t, t_0, b) = u(t, t_1, u(t_1, t_0, b)), \quad t, t_1 \in \mathbb{R}, b \in \mathbb{R}^n. \tag{2.7}
$$

Thus, a special solution is determined by its value at e.g. $t = 0$. Consequently, there are exactly n linearly independent special solutions of (2.1) up to a shift in time. As in [2, 3], we define the special matrix solution as follows.

Definition 2.2. The **special matrix solution** of (2.1) is the $n \times n$ matrix-valued function $X(t)$, $t \in \mathbb{R}$, whose columns are special solutions of (2.1) and such that $X(0) = I$, where I is the $n \times n$ identity matrix.

Theorem 2.3. *Assume (2.2)-(2.3), and let $X(t)$ be the special matrix solution of (2.1). Then:*

(i) For each $t \in \mathbb{R}$ the matrix $X(t)$ is non-singular. Moreover, $u(t, t_0, b) = X(t) X^{-1}(t_0) b$ for all $t, t_0 \in \mathbb{R}, b \in \mathbb{R}^n$, and

$$
\|X(t) X^{-1}(t_0)\| \le e^{\lambda_0 (t-t_0)}, \qquad t, t_0 \in \mathbb{R},\ t \le t_0, \tag{2.8}
$$

where λ_0 is as in Lemma 2.2.

(ii) For each $\phi \in C$, the solution $x(t) = x(\phi)(t)$ of (2.1) on $[0, \infty)$ with initial condition $x_0 = \phi$ satisfies

$$
x(t) = X(t)[\ell(\phi) + o(1)] \qquad \text{as} \quad t \to \infty, \tag{2.9}
$$

i.e., the limit

$$\lim_{t\to\infty} X^{-1}(t)x(t) =: \ell(\phi)$$

exists in $\mathbb{R}^n$*.*

Proof. The first statement follows from the definition of $X(t)$, Lemma 2.2 and (2.7). The proof of (ii) is analogous to the proof of [3, Theorem 4], using Lemma 2.2 above instead of [3, Theorem 2], and is omitted. ∎

Remark 2.1. The special matrix solution $X(t)$ satisfies (2.8) as in [3]; however here $\lambda_0 \in (-1/r, 0)$ is defined by (2.5), instead of $(\sum_{i=1}^{\infty} m_i + l)e^{-\lambda_0 r} = -\lambda_0$ (cf. [3, Eq. (10)]).

We now recall the adjoint theory for linear FDEs, and refer the reader to [9, Chapters 6 and 7] for general results and notation.

Let

$$\tilde{L}(t)\phi = \sum_{i=1}^{\infty} M_i(t)\phi(-\tau_i(t)) + L(t)\phi = \int_{-r}^{0} d_\theta \tilde{\eta}(t,\theta)\phi(\theta),$$

where $\tilde{\eta}(t,\theta)$ is an $n\times n$ matrix-valued function measurable in $\mathbb{R}\times\mathbb{R}$, with $\tilde{\eta}(t,\cdot)$ continuos from the left on $(-r,0)$, of bounded variation on $[-r,0]$ for all $t\in\mathbb{R}$, and normalized so that

$$\tilde{\eta}(t,\theta) = 0 \quad \text{for} \quad \theta \geq 0, \qquad \tilde{\eta}(t,\theta) = \tilde{\eta}(t,-r) \quad \text{for} \quad \theta \leq -r.$$

As usual, for $\mathbb{R}^{n*}$ the n-dimensional vector space of row vectors, define $C^* = C([0,r];\mathbb{R}^{n*})$, and consider the formal adjoint equation to (2.1) in C^*, given by

$$y(t) + \int_t^{t_0} y(s)\tilde{\eta}(s,t-s)ds = \text{constant} \qquad \text{for} \quad t \leq t_0 - r. \tag{2.10}$$

We also consider the formal duality ([2, 9]),

$$<\psi,\phi,t>= \psi(0)\phi(0) + \int_0^r \psi(\xi)\Big(\int_{-r}^{0} d_\beta[\tilde{\eta}(t+\xi,\beta-\xi)]\phi(\beta)\Big)d\xi, \qquad \psi\in C^*, \phi\in C, t\in\mathbb{R}.$$

Under conditions (2.2)-(2.3) with $m_i = 0$, $i\in\mathbb{N}$, i.e., for equations $\dot{x}(t) = L(t)x_t$, Arino and Pituk [2] determined explicitly the asymptotic representation of all its solutions, by using the formal adjoint theory to evaluate $\ell(\phi)$. In order to get this representation, the authors first proved the existence of a special matrix solution $Y(t)$ for the adjoint equation (2.10), with properties similar to the ones satisfied by $X(t)$. Then, the formal adjoint duality was used to express the limit $\ell(\phi)$ in (2.9) in terms of $X(t), Y(t)$ and ϕ (cf. [2, Theorems 3.1 and 4.1]).

The next theorem states that the results in [2] are valid for the present situation. The proofs are omitted, since they follow by adapting the arguments in Sections 3 and 4 of [2]. However, in order to adjust the proofs of Lemmas 3.1 and 4.2 in [2], we should use the above definition of λ_0, and arguments as in the proofs of Theorem 2.1 and Lemma 2.2.

Theorem 2.4. *Assume (2.2)-(2.3). Then, there exists a unique $n \times n$ matrix-valued function $Y(t), t \in \mathbb{R}$, called the* **special matrix solution** *of (2.10), locally of bounded variation on $\mathbb{R}$ and satisfying the following conditions:*

(i) for all $s \in \mathbb{R}$, each row of $Y(s)$ satisfies the formal adjoint equation (2.10);

(ii) $< Y^t, X_t, t >= I,\ t \in \mathbb{R}$;

(iii) $\|Y(t)\|e^{\lambda_0 t}$ is bounded for $t \geq 0$;

(iv) for each $\phi \in C$, the solution $x(t) = x(\phi)(t)$ of (2.1) with initial condition $x_0 = \phi$ satisfies

$$\ell(\phi) = \lim_{t\to\infty} X^{-1}(t)x(t) =< Y^0, \phi, 0 > . \tag{2.11}$$

In the above statement, and following the usual notation, for each $t \in \mathbb{R}$, the matrices X_t and Y^t are defined by $X_t(\theta) = X(t+\theta), -r \leq \theta \leq 0$, and $Y^t(s) = Y(t+s), 0 \leq s \leq r$. The columns of X_t are elements of C, and the rows of Y^t are elements of C^*.

To this end, we point out the relationship between special solutions, and the evolutionary system on C defined by the solutions of (2.1) [9]. Let $T(t,\sigma) : C \to C$ be the solution operator of (2.1), given by $T(t,\sigma)(\phi) = x_t(\sigma,\phi)$ for $t \geq \sigma, \phi \in C$, where $x_t(\sigma,\phi)$ is the solution of (2.1) with initial condition $x_\sigma = \phi$. Assume (2.2)-(2.3), and define $P(\sigma) = \mathit{span}\, X_\sigma = \{\phi \in C : \phi = X_\sigma c$ for some $c \in \mathbb{R}^n\}$, $Q(\sigma) = \{\phi \in C :< Y^\sigma, \phi, \sigma >= 0\}$. In the finite dimensional space $P(\sigma)$, $T(t,\sigma)$ is defined for all $t \in \mathbb{R}$. The asymptotic behaviour of solutions of (2.1) was described in [2] as follows. For all $\sigma \in \mathbb{R}$, the phase space is decomposed as $C = P(\sigma) \oplus Q(\sigma)$. For $t, \sigma \in \mathbb{R}$ and $\phi = \phi^{P(\sigma)} + \phi^{Q(\sigma)} \in C$, with $\phi^{P(\sigma)} \in P(\sigma), \phi^{Q(\sigma)} \in Q(\sigma)$, then $x_t(\sigma,\phi) = x_t(\sigma,\phi^{P(\sigma)}) + x_t(\sigma,\phi^{Q(\sigma)})$, and there exists $K > 0$ such that

$$\begin{aligned} &\|x_t(\sigma,\phi^{P(\sigma)})\| \leq Ke^{\lambda_0(t-\sigma)}\|\phi^{P(\sigma)}\|, \quad \phi^{P(\sigma)} \in P(\sigma), t \leq \sigma \\ &\|x_t(\sigma,\phi^{Q(\sigma)})\| \leq Ke^{(\sigma-t)/r}\|\phi^{Q(\sigma)}\|, \quad \phi^{Q(\sigma)} \in Q(\sigma), t \geq \sigma. \end{aligned} \tag{2.12}$$

Note that (2.12) establishes an asymptotic behaviour for solutions of (2.1) similar to the one for autonomous linear FDEs, which can be used to study perturbed linear equations (cf. [9]). In fact, for the autonomous case $\dot{x}(t) = \tilde{L}x_t$, clearly there is a relationship between its special solutions and its characteristic values, as shown in more detail in Section 3. For the non-autonomous case (2.1), (2.12) asserts that special solutions play a role similar to the one played by solutions of exponential form of autonomous equations.

Example 2.1. Consider the scalar FDE

$$\dot{x}(t) = \sum_{i=1}^{\infty} \alpha_i(t)x(t-\tau_i(t)), \tag{2.13}$$

where $\alpha_i, \tau_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, $|\alpha_i(t)| \leq m_i$, $0 \leq \tau_i(t) \leq r_i \leq r$ for $t \in \mathbb{R}$, $i \in \mathbb{N}$ and

$$\sum_{i=1}^{\infty} m_i e^{r_i/r} < 1/r. \tag{2.14}$$

Eq. (2.13) with $\alpha_i(t) \leq 0$ was considered by several authors (see [10] and references therein). From Theorems 2.1 and 2.3, there is a unique special solution $u(t)$ of (2.13) such that $u(0) = 1$, and any other solution $x(t) = x(\phi)(t)$ of (2.13) satisfies (2.9), where $X(t) = u(t)$. In particular, the equilibrium zero of (2.13) is globally attractive if and only if $u(t) \to 0$ as $t \to \infty$. If this is the case, the zero solution is exponentially asymptotically stable [9]. We remark that the uniform stability of the zero solution of (2.13) was proven in [10] for the situation where the above functions $\alpha_i(t)$ are all non-positive and under the assumption $\sum_{i=1}^{\infty} m_i r_i \leq 1$. Clearly, this latter condition is satisfied if (2.14) holds.

Example 2.2. Consider the FDE in $C([-\tau, 0]; \mathbb{R}^n)$

$$\dot{x}(t) = (\tilde{A} + A(t))x(t) + (\tilde{B} + B(t))x(t - \tau), \tag{2.15}$$

where $\tau > 0$, $\tilde{A} = diag\,(\tilde{a}_1, \ldots, \tilde{a}_n)$, $\tilde{B} = diag\,(\tilde{b}_1, \ldots, \tilde{b}_n)$ are constant diagonal matrices, $A = (a_{ij}), B = (b_{ij})$ are $n \times n$ matrix-valued continuous functions on $t \in [0, \infty)$. System (2.15) can be interpreted as a perturbation of the autonomous delayed system

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}x(t - \tau). \tag{2.16}$$

The aymptotic behaviour of solutions of the perturbed system (2.15) was studied in [1], by using the properties of the special matrix solution for the autonomous system (2.16). In fact, for $A(t), B(t)$ "small" and assuming that

$$\sup_{1 \leq i \leq n} \Big(|\tilde{a}_i| + |\tilde{b}_i| \Big) e < 1/\tau, \tag{2.17}$$

in [1, Theorem 3.1] the authors gave an asymptotic formula for the solutions of (2.15), extending previous results established in [8] for the scalar case (see further references in [1]). Condition (2.17) was imposed in order to have $(\|\tilde{A}\| + \|\tilde{B}\|)e < 1/r$, so that the results in [3] assured the existence of an $n \times n$ special matrix solution for (2.16) with the properties in Theorem 2.3. From the approach here, it is clear that the results in [1] follow under the weaker condition (2.3), which reads here as

$$\sup_{1 \leq i \leq n} \Big(|\tilde{a}_i| + |\tilde{b}_i| e \Big) < 1/\tau.$$

3. Characteristic values and special solutions of autonomous linear FDEs in $\mathbb{R}^n$

Consider the case of (2.1) autonomous,

$$\dot{x}(t) = \sum_{i=1}^{\infty} M_i x(t - r_i) + L x_t, \tag{3.1}$$

where $0 \le r_i \le r$ and M_i are $n \times n$ constant matrices for $i \in \mathbb{N}$, $L \in \mathcal{L}(C, \mathbb{R}^n)$ and $\sum_{i=1}^{\infty} \|M_i\| < \infty$.

In this setting, hypotheses (2.2)-(2.3) translate simply as

$$\sum_{i=1}^{\infty} \|M_i\| e^{r_i/r} + \|L\| e < \frac{1}{r}. \tag{3.2}$$

The constant $\lambda_0 \in (-1/r, 0)$ in Lemma 2.2 is defined by the relationship

$$\sum_{i=1}^{\infty} \|M_i\| e^{-\lambda_0 r_i} + \|L\| e^{-\lambda_0 r} = -\lambda_0. \tag{3.3}$$

Consider the C_0-semigroup $(T(t))_{t \ge 0}$ of bounded linear operators on C associated with the semiflow of (3.1),

$$T(t)\phi = x_t(\phi) \quad \text{for} \quad t \ge 0, \phi \in C,$$

where $x_t(\phi)$ is the solution of (3.1) with initial condition $x_0 = \phi$. Assume (3.2), and let $X(t)$ and $Y(t)$ be the special matrix solutions for (3.1) and its formal adjoint equation, respectively. Since (3.1) is autonomous, we have $X(t+s) = X(t)X(s)$ for $t, s \in \mathbb{R}$. In particular, $X^{-1}(t) = X(-t)$ and $T(t)X_0 = X_t = X_0 X(t)$, $t \in \mathbb{R}$. Hence, *span* $X_t =$ *span* X_0, where for a matrix $[\phi_1 \cdots \phi_n]$ with $\phi_1, \ldots, \phi_n \in C$ we denote by *span* $[\phi_1 \cdots \phi_n]$ the span of the set $\{\phi_1, \ldots, \phi_n\}$. Similar properties hold for $Y(t)$. From the results in the previous section, the phase space C is decomposed as the topological direct sum

$$C = P \oplus Q, \tag{3.4}$$

where $P := P(0) = \textit{span}\ X_0 = \{\phi \in C : \phi = X_0 c \text{ for some } c \in \mathbb{R}^n\}$, $Q := Q(0) = \{\phi \in C : < Y^0, \phi >= 0\}$, where $< \cdot, \cdot >$ is the formal adjoint duality, and P, Q are invariant spaces under $T(t), t \ge 0$. For $\phi = \phi^P + \phi^Q \in C$ decomposed according to (3.4), then $\phi^P = X_0 < Y^0, \phi >$, and

$$x_t(\phi) = T(t)\phi^P + T(t)\phi^Q = X_t < Y^0, \phi > + x_t(\phi^Q), \quad t \ge 0. \tag{3.5}$$

Therefore, the limit $\ell(\phi)$ in (2.9) is $\ell(\phi) =< Y^0, \phi >$, and from (2.12) there is $K > 0$ such that

$$\|x_t(\phi^P)\| \le K e^{\lambda_0 t} \|\phi^P\|, \qquad t \le 0,\ \phi^P \in P, \tag{3.6}$$

$$\|x_t(\phi^Q)\| \le K e^{-t/r} \|\phi^Q\|, \qquad t \ge 0,\ \phi^Q \in Q. \tag{3.7}$$

On the other hand, let A be the infinitesimal generator for $(T(t))_{t\geq 0}$. It is well-known [9] that A has only the point spectrum, $\sigma(A)=\sigma_P(A)$, and that $\lambda\in\mathbb{C}$ is an eigenvalue of A if and only if λ is a characteristic value for (3.1), i.e., λ is a root of the characteristic equation

$$\det\Delta(\lambda)=0,\quad \text{where}\quad \Delta(\lambda)=\sum_{i=1}^{\infty}M_ie^{-\lambda r_i}+L(e^{\lambda\cdot}I)-\lambda I, \tag{3.8}$$

for I the $n\times n$ identity matrix. For each $\lambda\in\sigma(A)$, denote by $\mathcal{M}_\lambda(A)$ the generalized eigenspace associated with λ. For a nonempty finite set $\Lambda\subset\sigma(A)$, let

$$C=P_\Lambda\oplus Q_\Lambda$$

be the decomposition of C by Λ according to the adjoint theory [9]. This means that $P_\Lambda=\oplus_{\lambda\in\Lambda}\mathcal{M}_\lambda(A)$ and $Q_\Lambda=\{\phi\in C:<\psi,\phi>=0 \text{ for every } \psi\in P_\Lambda^*\}$, with P_Λ^* the dual space of P_Λ, $P_\Lambda^*=\oplus_{\lambda\in\Lambda}\mathcal{M}_\lambda(A^*)$, and A^* the formal adjoint of A.

From the asymptotic behaviour of solutions described by the exponential dichotomy (3.6)-(3.7), it is clear that (3.2) imposes severe restrictions on the set of characteristic values of (3.1).

Theorem 3.1. *Assume (3.2), and let $\lambda_0\in(-1/r,0)$ be defined by (3.3). With the above notations,*

(i) if $\lambda\in\sigma(A)$ and $Re\,\lambda\geq-1/r$, then $|\lambda|<\frac{1}{r}$.

(ii) there are no eigenvalues λ of (3.1) in the strip $-1/r\leq Re\,\lambda<\lambda_0$;

(iii) if there is $\lambda\in\sigma(A)$ with $Re\,\lambda=\lambda_0$, then $\mathcal{M}_\lambda(A)=N(A-\lambda I)$, i.e., the ascent of the operator $A-\lambda I$ is one.

Proof. Let $\lambda\in\sigma(A)$ and suppose that $Re\,\lambda\geq-1/r$. From (3.8), we obtain that there is $b\in\mathbb{C}^n, b\neq 0$, such that $|\lambda||b|\leq(\sum_{i=1}^{\infty}\|M_i\|e^{r_i/r}+\|L\|e)|b|$, thus $|\lambda|<\frac{1}{r}$. In particular, $Re\,\lambda>-1/r$, and $x(t)=e^{\lambda t}b$ is a special solution of (3.1). From (3.6), we conclude now (ii).

Now consider $\lambda\in\sigma(A)$ with $Re\,\lambda=\lambda_0$. Let $\tilde{L}\phi:=\sum_{i=1}^{\infty}M_i\phi(-r_i)+L\phi$, and denote by $\tilde{L}_\lambda^1$ the $n\times n$ matrix defined by $\tilde{L}_\lambda^1x=\tilde{L}(e^{\lambda\theta}\theta x)$, where we abuse the notation and write $\tilde{L}(\phi(\theta))$ for $\tilde{L}\phi,\phi\in C$. Note that $\Delta'(\lambda)=\tilde{L}_\lambda^1-I$.

In order to get a contradiction, suppose that $\mathcal{M}_\lambda(A)=N(A-\lambda I)^d$, where $d>1$ is the ascent of $A-\lambda I$. Then (see [9]) there exists $\phi\in\mathcal{M}_\lambda(A)$ of the form

$$\phi(\theta)=e^{\lambda\theta}(u_0+\theta u_1),$$

with the vectors $u_0,u_1\in\mathbb{C}^n$, $u_1\neq 0$, satisfying

$$\Delta(\lambda)u_1=0,\quad \Delta(\lambda)u_0+(\tilde{L}_\lambda^1-I)u_1=0.$$

Clearly $u(t) = e^{\lambda t}(u_0 + tu_1)$ is a solution of (3.1) on $\mathbb{R}$ and $|u(t)|e^{t/r} = e^{(\lambda_0+1/r)t}|u_0 + tu_1| \to 0$ as $t \to -\infty$. Therefore, there is $c \in \mathbb{C}^n$ such that $u(t) = X(t)c$. For $t \leq 0$, we get $|u(t)| = e^{\lambda_0 t}|u_0 + tu_1| \leq e^{\lambda_0 t}|c|$, a contradiction since $u_1 \neq 0$. This proves (iii). ∎

Write (3.1) as $\dot{x}(t) = \tilde{L}x_t$, where $\tilde{L}$ is as before. By the change of variables $y(t) = e^{t/r}x(t)$, (3.1) is transformed into

$$\dot{y}(t) = \frac{1}{r}y(t) + \tilde{L}(e^{-\frac{1}{r}\cdot}y_t), \tag{3.9}$$

and λ is a characteristic value for (3.1) if and only if $\lambda + \frac{1}{r}$ is a characteristic value for (3.9). Thus, $x(t)$ is a special solution of (3.1) if and only if $e^{t/r}x(t)$ is a solution of (3.9) defined on $\mathbb{R}$ and bounded for $t \leq 0$. From Theorem 3.1 and the definition of unstable space for an autonomous linear FDE on $\mathbb{R}^n$, we deduce the following result:

Theorem 3.2. *Assume (3.2). Then, there are exactly n characteristic values λ of (3.1) (counting multiplicities) with $Re\,\lambda \geq -1/r$. Moreover, with the above notations, $P = P_\Lambda$ and $Q = Q_\Lambda$, where $\Lambda = \{\lambda \in \sigma(A) : Re\,\lambda \geq -1/r\} = \{\lambda \in \sigma(A) : Re\,\lambda \geq \lambda_0\}$.*

Consequently, an explicit representation of the special matrix solution $X(t)$ can be given in terms of a basis of P_Λ. Consider $\Phi_\Lambda, \Psi_\Lambda$ normalized dual bases for P_Λ, P_Λ^*, i.e., $P_\Lambda = span\,\Phi_\Lambda, P_\Lambda^* = span\,\Psi_\Lambda$ and $< \Psi_\Lambda, \Phi_\Lambda >= I$, and B_Λ the $n \times n$ matrix such that $\dot{\Phi}_\Lambda = \Phi_\Lambda B_\Lambda, -\dot{\Psi}_\Lambda = B_\Lambda \Psi_\Lambda$. Then $\Phi_\Lambda = X_0\Phi_\Lambda(0)$, i.e., $\Phi_\Lambda(\theta) = X(\theta)\Phi_\Lambda(0), -r \leq \theta \leq 0$. It follows that $X(t) = \Phi_\Lambda(0)e^{B_\Lambda t}\Phi_\Lambda^{-1}(0)$ and $T(t)\Phi_\Lambda(0) = \Phi_\Lambda(0)e^{B_\Lambda t} = X(t)\Phi_\Lambda(0), t \in \mathbb{R}$. For $\phi \in C$, the solution $x(t) = x_t(\phi)(0)$ of (3.1) with initial condition $x_0 = \phi$ is given by

$$x_t(\phi) = T(t)\Phi_\Lambda < \Psi_\Lambda, \phi > + x_t(\phi^Q),$$

for all $t \geq 0$ and $\phi = \phi^P + \phi^Q$ decomposed by (3.4), with $T(t)\Phi_\Lambda = \Phi_\Lambda e^{B_\Lambda t}$. We have

$$\ell(\phi) := \lim_{t\to\infty} X^{-1}(t)x(\phi)(t) = \phi^P(0), \tag{3.10}$$

with $\phi^P(0) = \Phi_\Lambda(0) < \Psi_\Lambda, \phi >$. From Theorem 3.1 and [9] we obtain that there are $K_1 > 0$ and $\delta > 0$ such that

$$\|x_t(\phi^Q)\| \leq K_1 e^{-(1/r+\delta)t}\|\phi^Q\|, \qquad t \geq 0,\ \phi^Q \in Q.$$

Hence $x_t(\phi^Q) = o(e^{-t/r})$ as $t \to \infty$, which is a better estimate than (3.7), and we write

$$x(\phi)(t) = X(t)\ell(\phi) + o(e^{-t/r}) \quad \text{as} \quad t \to \infty. \tag{3.11}$$

Remark 3.1. Theorems 3.1 and 3.2 complete and generalize the conclusions in [2] for the particular case of equation $\dot{x}(t) = Bx(t-r)$, where B is an $n \times n$ constant matrix.

Example 3.1. Consider a scalar linear FDE on $C = C([-r, 0]; \mathbb{R})$,

$$\dot{x}(t) = \sum_{i=1}^{\infty} a_i x(t - r_i) + Lx_t, \tag{3.12}$$

with $0 \le r_i \le r,\ a_i \in \mathbb{R}, i = 1, 2, \dots,\ L : C \to \mathbb{R}$ a linear bounded operator, and assume (3.2):

$$\sum_{i=1}^{\infty} |a_i| e^{r_i/r} + \|L\| e < 1/r. \tag{3.13}$$

For related examples see [5]. The characteristic equation for (3.12) is

$$\Delta(\lambda) := \sum_{i=1}^{\infty} a_i e^{-\lambda r_i} + L(e^{\lambda \cdot}) - \lambda = 0.$$

From (3.13), we have $\Delta(-1/r) \ge \sum_{i=1}^{\infty} a_i e^{r_i/r} - \|L\| e + 1/r > 0$. Since $\Delta(\lambda) \to -\infty$ as $\lambda \to \infty$ ($\lambda \in \mathbb{R}$), we deduce that there is a real root $\lambda_* \in (-1/r, \infty)$ of the characteristic equation.

We now apply the above results. Since (3.13) holds, there is a unique special solution (up to a scalar multiplication or shift in time) of (3.12). From Theorem 3.2, we conclude then that the set $\Lambda = \{\lambda \in \sigma(A) : Re\, \lambda \ge -1/r\}$ reduces to $\Lambda = \{\lambda_*\}$, with λ_* a simple root and $\lambda_* \in [\lambda_0, 1/r)$, where $\lambda_0 \in (-1/r, 0)$ satisfies (3.3) with $\|M_i\| = |a_i|$. With the above notations, $X(t) = e^{\lambda_* t}, Y(t) = e^{-\lambda_* t}$, $P = P_\Lambda = span\,\{e^{\lambda_* \cdot}\}$. From (3.5) and (3.11), we derive that, for each $\phi \in C$, the solution $x(\phi)$ of (3.12) with initial condition $x_0 = \phi$ satisfies

$$x(\phi)(t) = e^{\lambda_* t} \ell(\phi) + o(e^{-t/r}) \quad \text{as} \quad t \to \infty,$$

with

$$\begin{aligned} \ell(\phi) &= < e^{-\lambda_* \cdot}, \phi > \\ &= \phi(0) + \sum_{i=1}^{\infty} a_i \int_{-r_i}^{0} e^{-\lambda_*(r_i + \xi)} \phi(\xi) d\xi + L\Big(\int_{\theta}^{0} e^{\lambda_*(\theta - \xi)} \phi(\xi) d\xi \Big). \end{aligned}$$

In particular,

$$x(\phi)(t) = O(e^{\lambda_* t}) \quad \text{as} \quad t \to \infty, \quad \text{for all } \phi \in C. \tag{3.14}$$

If in addition

$$\Delta(0) = \sum_{i=1}^{\infty} a_i + L(1) < 0,$$

then $\lambda_* < 0$, and the trivial solution of (3.12) is exponentially asymptotically stable. For the particular case

$$\dot{x}(t) = ax(t) + Lx_t, \tag{3.15}$$

with $a \in \mathbb{R}$ and $L : C \to \mathbb{R}$ a linear bounded operator, $x = 0$ is exponentially asymptotically stable if $|a| + \|L\|e < 1/r$ and $a + L(1) < 0$.

4. Special solutions of autonomous linear FDEs in Banach spaces

In this section, we consider linear autonomous FDEs in Banach spaces, and use special solutions to study the asymptotic behaviour of all solutions.

Let $\mathcal{X}$ be a Banach space, $r > 0$, and $C = C([-r, 0]; \mathcal{X})$ the space of continuous functions from $[-r, 0]$ to $\mathcal{X}$ with the norm $\|\phi\| = \max_{-r\le\theta\le 0} |\phi(\theta)|, \phi \in C$, where $|\cdot|$ is the norm in $\mathcal{X}$. In the phase space C, we consider a linear autonomous FDE of the form

$$\dot{u}(t) = A_T u(t) + \sum_{i=1}^{\infty} B_i u(t - r_i) + Lu_t\,, \tag{4.1}$$

where $A_T : D(A_T) \subset \mathcal{X} \to \mathcal{X}$ is a linear operator, $B_i \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ $(i \in \mathbb{N}), \sum_{i=1}^{\infty} \|B_i\| < \infty$, and $L \in \mathcal{L}(C, \mathcal{X})$. As usual, we write $u_t \in C$ for $u_t(\theta) = u(t + \theta), -r \le \theta \le 0$, and denote by $u(\phi)(t)$ the solution of (4.1) with initial condition $u_0 = \phi$.

We are particularly interested in equations (4.1), since they include reaction-diffusion equations with delays appearing in the reaction terms. Equations involving both time delays and spatial diffusion have been increasingly used in population dynamics, disease modelling, and other fields. For instance, for $\Omega \subset \mathbb{R}^n$ open, choose $\mathcal{X}$ as an appropriate Banach space of functions from $\overline{\Omega}$ to $\mathbb{R}^m$. Typically, a linear delayed reaction-diffusion equation in Ω takes the form (4.1), where a diffusion term $d\Delta v(t, x), d = (d_1, \ldots, d_n) \in \mathbb{R}^n$ constant, is given by $A_T u(t) = d\Delta v(t, x)$, for $u(t)(x) := v(t, x), x \in \Omega$, with $\mathrm{dom}(\Delta) \subset \mathcal{X}$, and the delayed reaction terms are given by an operator in $\mathcal{L}(C; \mathcal{X})$. A very simple example of a delayed reaction-diffusion model is Hutchinson's equation with diffusion, which arises from a generalization of the delayed logistic equation, $\dot{y}(t) = ay(t)[1-y(t-r)/K]$. Translating the equilibrium K to the origin by the change $v(t) = -1+y(t)/K$, this equation becomes $\dot{v}(t) = -av(t - r)[1 + v(t)]$. Considering also a spatial variable $x \in [0, \pi]$ and diffusion terms, we obtain the model

$$\frac{\partial v}{\partial t}(t, x) = d\frac{\partial^2 v}{\partial x^2}(t, x) - av(t - r, x)[1 + v(t, x)], \quad t > 0, x \in (0, \pi),$$

(where d is the diffusion rate), to which some boundary conditions should be added. Clearly its linearization at the origin, which will be considered in Example 4.1, has the form (4.1).

We now assume some restrictions on the operators A_T, B_i ($i \in \mathbb{N}$) and L in (4.1), in order to use the results for linear FDEs in $\mathbb{R}^n$ in Section 3. Roughly speaking, we shall assume that the Banach space $\mathcal{X}$ can be decomposed by eigenspaces for A_T, and that the operators B_i and L do not mix the spatial variations described by such eigenspaces. This latter condition is very restrictive, since it almost imposes that B_i and L are scalar multiplications. However, the framework established below is still useful in terms of applications.

For A_T, B_i ($i \in \mathbb{N}$) and L, we shall assume the following assumptions (for similar assumptions, cf. [14, p. 82] and references therein):

(H1) A_T generates a compact C_0-semigroup of linear operators $\{T(t)\}_{t\geq 0}$ on $\mathcal{X}$;

(H2) the point spectrum $\sigma_P(A_T)$ of A_T consists of a sequence of (distinct) real numbers $\{\mu_k\}_{k=1}^{\infty}$; moreover, each $x \in D(A_T)$ can be written in a unique way as $x = \sum_{k=1}^{\infty} P_k x$, with $A_T x = \sum_{k=1}^{\infty} \mu_k P_k x$, where $P_k : \mathcal{X} \to F_k$ is the canonical projection from $\mathcal{X}$ onto $F_k := Ker\,(A_T - \mu_k I)$, and is continuous, $k \in \mathbb{N}$;

(H3) $B_i P_k = P_k B_i$ for $i, k \in \mathbb{N}$;

(H4) if $E_k := \{\phi \in C : \phi(\theta) \in F_k,\ \theta \in [-r, 0]\}$, then $L(E_k) \subset F_k$, for all $k \in \mathbb{N}$.

Remark 4.1. Recall that (H1) implies that the spectrum of A_T consists of a sequence of eigenvalues $\{\mu_k\}$ with $-\infty$ as the only possible accumulation point [11, p. 51]. On the other hand, from (H2), it follows that $F_k = Ker\,(A_T - \mu_k I) = \mathcal{M}_{\mu_k}(A_T)$, i.e., the ascent of the operators $A_T - \mu_k I$ (where I is the identity operator on $\mathcal{X}$) is one, $k \in \mathbb{N}$.

In the sequel, given $p \in \mathbb{N}$ we write C_p for the space $C([-r, 0]; \mathbb{R}^p)$ equipped with the sup norm. For each $k \in \mathbb{N}$, let $d_k = \dim F_k$, and $F_k = span\,\{\beta_{k,1}, \dots, \beta_{k,d_k}\}$. Under (H1)–(H4), the characteristic equation

$$\Delta(\lambda)x := A_T x + \sum_{i=1}^{\infty} e^{-\lambda r_i} B_i x + L(e^{\lambda \cdot} x) - \lambda x = 0, \quad x \in D(A_T), \tag{4.2}$$

is decomposed into a sequence of characteristic equations ([14])

$$\det \Delta_k(\lambda) = 0, \quad k \in \mathbb{N}, \tag{4.3$_k$}$$

with the $d_k \times d_k$ matrix $\Delta_k(\lambda)$ defined by

$$\Delta_k(\lambda) = M_k + \sum_{i=1}^{\infty} e^{-\lambda r_i} B_{i,k} + L_k(e^{\lambda \cdot} I_{d_k}) - \lambda I_{d_k}, \quad k \in \mathbb{N},$$

where M_k is the $d_k \times d_k$ matrix $M_k = diag\,(\mu_k, \dots, \mu_k)$, $B_{i,k}$ are $d_k \times d_k$ matrices and $L_k : C_{d_k} \to \mathbb{R}^{d_k}$ is a linear operator, given by

$$B_i\Big(\sum_{j=1}^{d_k} x_{k,j}\beta_{k,j}\Big) = \Big(B_{i,k}\begin{pmatrix} x_{k,1} \\ \vdots \\ x_{k,d_k}\end{pmatrix}\Big)^{\top}\begin{pmatrix} \beta_{k,1} \\ \vdots \\ \beta_{k,d_k}\end{pmatrix}$$

$$L\Big(\sum_{j=1}^{d_k} \psi_{k,j}\beta_{k,j}\Big) = \Big(L_k\begin{pmatrix} \psi_{k,1} \\ \vdots \\ \psi_{k,d_k}\end{pmatrix}\Big)^{\top}\begin{pmatrix} \beta_{k,1} \\ \vdots \\ \beta_{k,d_k}\end{pmatrix},$$

for $x_k = (x_{k,1}, \dots, x_{k,d_k}) \in \mathbb{R}^{d_k}$, $\psi_k = (\psi_{k,1}, \dots, \psi_{k,d_k}) \in C_{d_k}$, $k \in \mathbb{N}$, where $^{\top}$ denotes the transpose of a matrix. In other words, $B_{i,k}$ is the restriction of B_i to F_k, and L_k is the restriction of L to $C([-r,0];F_k)$, with values in F_k, if F_k is identified with $\mathbb{R}^{d_k}$. Note that for each $k \in \mathbb{N}$, (4.3$_k$) is the characteristic equation for the FDE in C_{d_k}

$$\dot{x}(t) = M_k x(t) + \sum_{i=1}^{\infty} B_{i,k} x(t - r_i) + L_k x_t. \tag{4.4$_k$}$$

In this setting, we look for special solutions of (4.1) when the delays are small. As for FDEs in $\mathbb{R}^n$, we say that $u(t)$ is a **special solution** of (4.1) if $u(t)$ is a solution of (4.1) defined on $\mathbb{R}$ and $|u(t)|e^{t/r}$ is bounded for $t \le 0$.

Suppose that

$$\mu_k + \gamma < 1/r, \quad \text{for all } k \in \mathbb{N}, \tag{4.5}$$

where

$$\gamma := \sum_{i=1}^{\infty} \|B_i\| e^{r_i/r} + \|L\| e\,.$$

Note that for reaction-diffusion equations with delays, typically A_T is the operator $d\Delta$ for some constant $d \in \mathbb{R}^n$, with domain in an appropriate Banach space of functions, and whose eigenvalues are non-positive. In this situation, (4.5) follows from $\gamma < 1/r$.

Let Σ_0 be the finite set

$$\Sigma_0 = \{\mu \in \sigma(A_T) : |\mu| < 1/r - \gamma\},$$

and suppose that $\Sigma_0 \neq \emptyset$. For the case $\Sigma_0 = \emptyset$, see Remark 4.2 below. Consider the eigenvalues of A_T ordered in such a way that

$$\Sigma_0 = \{\mu_1, \dots, \mu_s\},$$

i.e, $\mu_1, \dots, \mu_s$ are exactly the eigenvalues μ_k of A_T that satisfy

$$|\mu_k| + \gamma < 1/r. \tag{4.6}$$

From (4.5), we may write $\sigma(A_T) = \Sigma_0 \cup \Sigma_1$, where

$$\Sigma_1 = \{\mu \in \sigma(A_T) : \mu \leq -1/r + \gamma\} = \{\mu_k : k \in \mathbb{N}, k > s\}.$$

Define

$$F^0 = F_1 \oplus \cdots \oplus F_s,$$

and P^0 the canonical projection induced by the projections $P_1, \ldots, P_k$,

$$P^0 : \mathcal{X} \to F^0, \quad P^0 x = \sum_{k=1}^{s} P_k x.$$

With $N := \dim F^0 = \sum_{1 \leq k \leq s} d_k$, F^0 can be identified with $\mathbb{R}^N$.

Consider now equations (4.4_k), $1 \leq k \leq s$, all together. In $C([-r, 0]; F^0) \equiv C([-r, 0]; \mathbb{R}^N) = C_N$, Eq. (4.1) reads as

$$\dot{x}(t) = \hat{B}_0 x(t) + \sum_{i=1}^{\infty} \hat{B}_i x(t - r_i) + \hat{L} x_t, \tag{4.7}$$

where $\hat{B}_0, \hat{B}_i$ are $N \times N$ matrices, $\hat{B}_0 = diag\,(M_1, \ldots, M_s)$, $\hat{B}_i = diag\,(B_{i,1}, \ldots, B_{i,s})$, $i = 1, 2, \ldots$, and $\hat{L} : C_N \to \mathbb{R}^N$ is the bounded linear operator defined by

$$\hat{L} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_s \end{pmatrix} = \begin{pmatrix} L_1(\phi_1) \\ \vdots \\ L_s(\phi_s) \end{pmatrix}, \qquad \text{for} \quad \phi_k \in C_{d_k}, 1 \leq k \leq s.$$

From (4.6), we have

$$\|\hat{B}_0\| + \sum_{i=1}^{\infty} \|\hat{B}_i\| e^{r_i/r} + \|\hat{L}\| e \leq \max_{1 \leq k \leq s} |\mu_k| + \gamma < 1/r. \tag{4.8}$$

This means that Eq. (4.7) is a particular case of Eq. (3.1) satisfying condition (3.2). From the material in Section 3, consider the special matrix solution $X(t)$ for (4.7), and the special matrix solution $Y(t)$ for its adjoint equation. Also, decompose C_N as

$$C_N = P \oplus Q, \tag{4.9}$$

where $P = span\ X_0$ and $Q = \{\phi \in C_N :< Y^0, \phi >= 0\}$.

We shall now address the asymptotic behaviour of all solutions of (4.1) in the whole space $C = C([-r, 0]; \mathcal{X})$, by imposing some more restrictions on the eigenvalues in Σ_1. In the next theorem, for all solutions of (4.1) we recover the asymptotic behaviour established in (3.11).

Theorem 4.1. *Assume (H1)-(H4) and (4.5). With the above notations, suppose also that* $\Sigma_0 = \{\mu_1, \dots, \mu_s\} \neq \emptyset$ *and*

$$\mu_k < -1/r - \gamma \quad \text{for all} \quad k > s, \tag{4.10}$$

i.e., there are no eigenvalues of A_T *on the interval* $[-1/r-\gamma, -1/r+\gamma]$*. Then, all solutions* $u(\phi)(t)$ *of (4.1) on* $[0, \infty)$ *satisfy*

$$u(\phi)(t) = X(t) < Y^0, \pi\phi > +o(e^{-t/r}) \quad \text{as} \quad t \to \infty, \tag{4.11}$$

where $\pi : C \to C_N$ *is the projection defined by* $(\pi\phi)(\theta) = P^0(\phi(\theta)), -r \leq \theta \leq 0$*, and* F^0 *is identified with* $\mathbb{R}^N$*.*

Proof. Let A be the infinitesimal generator of the C_0-semigroup $(U(t))_{t\geq 0}$ generated by the semiflow of (4.1), $U(t)\phi = u_t(\phi)$, where now $u(t) = u(\phi)(t)$ is the mild solution of (4.1) on $[-r, \infty)$ with initial condition $u_0 = \phi \in C$ [13, 14]. The eigenvalues of A are exactly the roots of the characteristic equation (4.2) and the spectrum of A is reduced to the point spectrum. Thus $\lambda \in \sigma(A)$ if and only if there is $k \in \mathbb{N}$ such that $\det \Delta_k(\lambda) = 0$. Define

$$\Lambda = \{\lambda \in \sigma(A) : Re\, \lambda \geq -1/r\}.$$

Now, let $\lambda \in \mathbb{C}$ with $\det \Delta_k(\lambda) = 0$ for some $k > s$, and suppose that $Re\, \lambda \geq -1/r$. The matrix M_k in (4.3$_k$) is given by $M_k = \mu_k I_{d_k}$, therefore λ has the form $\lambda = \mu_k + \sigma_k(\lambda)$, where $\sigma_k(\lambda)$ is an eigenvalue of the matrix $\sum_{i=1}^{\infty} e^{-\lambda r_i} B_{i,k} + L_k(e^{\lambda \cdot} I_{d_k})$. Assumption $Re\, \lambda \geq -1/r$ implies that

$$|\sigma_k(\lambda)| \leq \sum_{i=1}^{\infty} e^{r_i/r} \|B_{i,k}\| + \|L_k\| e \leq \gamma,$$

and from (4.10) we obtain $Re\, \lambda < -1/r$, a contradiction. This proves that $\Lambda = \{\lambda \in \mathbb{C} : \det \Delta_k(\lambda) = 0, \text{ for some } k = 1, \dots, s\}$.

Since P^0 is a continuous projection, $\pi : C \to C_N$ defined as above is also a continuous projection. Let $\mathcal{X}$ and C be decomposed as

$$\mathcal{X} = F^0 \oplus F^1, \quad C = C_N \oplus C_{F^1},$$

where $F^1 = Ker\, P^0$ and $C_{F^1} = Ker\, \pi = C([-r, 0]; F^1)$. For the solution operator $U(t), t \geq 0$, of (4.1), from (H1)-(H4) it follows that the closed subspaces C_N, C_{F^1} are invariant under $U(t)$. Thus

$$u_t(\phi) = u_t(\pi\phi) + u_t((I - \pi)\phi), \tag{4.12}$$

where $u_t(\pi\phi) = \pi(u_t(\phi)) = x_t(\pi\phi)$ is the solution of (4.7) with initial condition $x_0 = \pi\phi$.

Choose $\delta > 0$ such that $Re\,\lambda < -(1/r + \delta)$ for all roots λ of the characteristic equations $(4.3_k), k > s$. Hence (cf. e.g. [14, p. 78]), for some $K > 0$,

$$\|u_t((I-\pi)\phi)\| \leq Ke^{-(1/r+\delta)t}\|(I-\pi)\phi\|, \quad t \geq 0,\ \phi \in C.$$

On the other hand, by using the decomposition $C_{F^0} \equiv C_N = P \oplus Q$ in (4.9) and the results in Section 3 applied to (4.7), we have

$$x(\pi\phi)(t) = x_t(\pi\phi)(0) = X(t) < Y^0, \pi\phi > + o(e^{-t/r}) \quad \text{as} \quad t \to \infty. \tag{4.13}$$

This completes the proof. ∎

Remark 4.2. As proven above, if (4.5) and (4.10) hold, then $\Lambda := \{\lambda \in \mathbb{C} : Re\,\lambda \geq -1/r\} = \cup_{1\leq k\leq s}\{\lambda \in \mathbb{C} : \det\Delta_k(\lambda) = 0\}$. For the case $\mu_k < -1/r - \gamma$ for all $k \in \mathbb{N}$, then $\Sigma_0 = \emptyset$. In this situation the above proof holds with $F^0 = \{0\}$ and $F^1 = C$, implying that $u(\phi)(t) = o(e^{-t/r})$ as $t \to \infty$ for all $\phi \in C$.

The next result states that, under some conditions, the solutions of (4.1) have the asymptotic behaviour established in (2.9) and (3.10).

Theorem 4.2. *Assume (H1)-(H4) and (4.5). With the above notations, suppose that* $\Sigma_0 = \{\mu_1, \dots, \mu_s\} \neq \emptyset$ *and*

$$\max_{1\leq k\leq s} |\mu_k| + 3\gamma < 1/r. \tag{4.14}$$

Then for all $\phi \in C$*, and with the identification* $\mathbb{R}^N \equiv F^0 \subset \mathcal{X}$*,*

$$u(\phi)(t) = X(t)[< Y^0, \pi\phi > + o(1)] \quad \text{as} \quad t \to \infty. \tag{4.15}$$

In other words, for all $\phi \in C$*, there exists*

$$\lim_{t\to\infty} X(-t)u(\phi)(t) = < Y^0, \pi\phi > .$$

Proof. Let $\phi \in C$. From (4.12) and (4.13), it is sufficient to prove that,

$$u(\phi)(t) = X(t)o(1) \quad \text{as} \quad t \to \infty, \qquad \text{for all} \quad \phi \in F^1.$$

Define $g(\alpha) = -1/r + \sum_{i=1}^{\infty} \|B_i\|(e^{r_i/r} + e^{-\alpha r_i}) + \|L\|(e + e^{-\alpha r}) - \alpha$ for $\alpha < 0$. Clearly $g(0) < 0, g(-\infty) = \infty$ and $g'(\alpha) < 0, \alpha < 0$, thus there is a unique root α_* of $g(\alpha) = 0$ on $(-\infty, 0)$.

Claim 1: All the roots of the characteristic equations (4.3_k), $k > s$, satisfy $Re\,\lambda \le \alpha_*$.

Let $\lambda \in \mathbb{C}$ with $\det \Delta_k(\lambda) = 0$ for some $k > s$. As in the proof of Theorem 4.1, λ has the form $\lambda = \mu_k + \sigma_k(\lambda)$, for some $\sigma_k(\lambda)$ an eigenvalue of the matrix $\sum_{i=1}^{\infty} e^{-\lambda r_i} B_{i,k} + L_k(e^{\lambda \cdot} I_{d_k})$. If $Re\,\lambda > \alpha_*$, then

$$|\sigma_k(\lambda)| \le \sum_{i=1}^{\infty} e^{-\alpha_* r_i} \|B_{i,k}\| + \|L_k\| e^{-\alpha_* r}.$$

Since $\mu_k \in \Sigma_1$, from (4.14), we obtain

$$\begin{aligned} \alpha_* < Re\,\lambda &\le -\frac{1}{r} + \gamma + \sum_{i=1}^{\infty} e^{-\alpha_* r_i} \|B_{i,k}\| + \|L_k\| e^{-\alpha_* r} \\ &\le -\frac{1}{r} + \sum_{i=1}^{\infty} (e^{r_i/r} + e^{-\alpha_* r_i}) \|B_i\| + \|L\|(e + e^{-\alpha_* r}) = g(\alpha_*) + \alpha_*, \end{aligned}$$

hence $g(\alpha_*) > 0$, a contradiction. Therefore, $Re\,\lambda \le \alpha_*$, and claim 1 is proven.

For Eq. (4.7), define $\lambda_0 \in (-1/r, 0)$ as in (3.3), i.e., λ_0 satisfies

$$\max_{1 \le k \le s} |\mu_k| + \sum_{i=1}^{\infty} \|\hat{B}_i\| e^{-\lambda_0 r_i} + \|\hat{L}\| e^{-\lambda_0 r} = -\lambda_0. \tag{4.16}$$

Claim 2: $\alpha_* < \lambda_0$.

It is clear that $\alpha_* < \lambda_0$ if and only if $g(\lambda_0) < 0$. From (4.16), we have

$$\begin{aligned} g(\lambda_0) &= -\frac{1}{r} + \sum_{i=1}^{\infty} (e^{r_i/r} + e^{-\lambda_0 r_i}) \|B_i\| + \|L\|(e + e^{-\lambda_0 r}) - \lambda_0 \\ &= \max_{1 \le k \le s} |\mu_k| - \frac{1}{r} + \gamma + \sum_{i=1}^{\infty} e^{-\lambda_0 r_i} (\|B_i\| + \|\hat{B}_i\|) + e^{-\lambda_0 r}(\|L\| + \|\hat{L}\|) \\ &\le \max_{1 \le k \le s} |\mu_k| - \frac{1}{r} + 3\gamma < 0. \end{aligned}$$

Claim 3: $u(\phi)(t) = X(t)o(1)$ as $t \to \infty$, for all $\phi \in F^1$.

From claims 1 and 2, we conclude ([9]) that there are $K > 0$ and $\delta > 0$ such that

$$\|u_t(\phi)\| \le K e^{(\lambda_0 - \delta)t} \|\phi\|, \quad t \ge 0, \phi \in F^1.$$

From Theorem 2.3, we have $\|X(-t)\| = \|X^{-1}(t)\| \le e^{-\lambda_0 t}, t \ge 0$, hence, for $\phi \in F^1$,

$$|X(-t)u(\phi)(t)| \le K e^{-\delta t} \|\phi\| \to 0 \quad \text{as} \quad t \to \infty.$$

This proves claim 3, and completes the proof of the theorem. ■

Remark 4.3. Assumptions (H2)-(H4) can be slightly weakened by imposing that the eigenvalues of A_T can be organized in blocks, in such a way that the modes of the generalized eigenspaces for A_T generated by the eigenvalues in each block are not mixed by B_i and L ($i \in \mathbb{N}$), provided that either all or none of the eigenvalues in the same block belong to the set Σ_0 defined by condition (4.6). To be more precise, suppose that the following assumption is satisfied:

(H2') the spectrum $\sigma_P(A_T)$ of A_T consists of a sequence of real numbers, organized in blocks of eigenvalues $\Gamma_k = \{\mu_k^{i_k} : i_k = 1, \ldots, p_k\}$, $k \in \mathbb{N}$; moreover, each $x \in D(A_T)$ can be written in a unique way as $x = \sum_{k=1}^{\infty} P_k x$, with $A_T x = \sum_{k=1}^{\infty} A_T P_k x$, where $P_k : \mathcal{X} \to F_k$ is the canonical projection from $\mathcal{X}$ onto $F_k := \oplus_{1 \leq i_k \leq p_k} Ker\,(A_T - \mu_k^{i_k} I)$, and is continuous, $k \in \mathbb{N}$.

Consider now (H3) and (H4) with P_k and F_k, $k \in \mathbb{N}$, defined as in (H2'). Then, it is clear that Theorems 4.1 and 4.2 are valid if we replace (H2) by (H2'), define $\Sigma_0 = \Gamma_1 \cup \ldots \cup \Gamma_s$, $\Sigma_1 = \cup_{k>s} \Gamma_s$, and assume that $\Sigma_0 = \{\mu \in \sigma(A_T) : |\mu| < 1/r - \gamma\}$.

Remark 4.4. It is of interest to study the general case of autonomous linear FDEs in Banach spaces of the form

$$\dot{u}(t) = A_T u(t) + \tilde{L} u_t \tag{4.17}$$

without imposing (H2)-(H4). For (4.17), assume only that (H1) holds and that $\tilde{L}\phi = \int_{-r}^{0} d\eta(\theta)\phi(\theta)$, $\phi \in C$, for some function $\eta : [-r, 0] \to \mathcal{L}(\mathcal{X}; \mathcal{X})$ of bounded variation. In this situation, a complete formal adjoint theory for linear FDEs (4.17) was developed in [4], by using the formal duality introduced in [13]. It seems then natural to use this adjoint theory to look for special solutions of (4.17), since it was shown in Section 3 that special solutions for equations (3.1) can be identified with solutions associated with the set of characteristic values $\Lambda = \{\lambda \in \sigma(A) : Re\,\lambda \geq -1/r\}$. However, in applications to linear FDEs in Banach spaces, this theoretical approach does not seem to provide much information. In fact, unless additional conditions on A_T and $\tilde{L}$ are given, in general it is particularly difficult to find the set Λ defined above, since the characteristic operator is no longer an $n \times n$ matrix. Also, this approach does not relate the number of linearly independent special solutions with the size of the delay. For these reasons, we have considered the more restrictive framework where (H1)-(H4) hold. Nevertheless, the adjoint theory in [4] seems to be a natural tool to further pursue the analysis of the general case (4.17).

Example 4.1. Consider the following linear partial FDE,

$$\frac{\partial v}{\partial t}(t, x) = d\frac{\partial^2 v}{\partial x^2}(t, x) - av(t - r, x), \quad t > 0, x \in (0, \pi), \tag{4.18}$$

where $a, d, r > 0$, subjected to Neumann conditions

$$\frac{\partial v}{\partial t}(t,0) = \frac{\partial v}{\partial t}(t,\pi) = 0, \ t > 0. \tag{4.19}$$

Note that (4.18) is the linearization at the origin of Hutchinson's equation with diffusion. Define the Banach space $\mathcal{X}$ as $\mathcal{X} = \{v \in W^{2,2}[0,\pi] : v'(0) = v'(\pi) = 0\}$ with norm $\|\cdot\|_{2,2}$ induced from the norm of $W^{2,2}$. For $u(t)(x) := v(t,x), x \in [0,\pi]$, (4.18)-(4.19) is written in abstract form as the linear FDE in $C = ([-1,0];\mathcal{X})$

$$\dot{u}(t) = d\Delta u(t) - au(t-r) \tag{4.20}$$

which has the form (4.1), with $A_T = d\Delta$ and $\tilde{L}(\phi) = -a\phi(-r)$. The eigenvalues of A_T are $\mu_k = -dk^2, k \geq 0$, with the corresponding normalized eigenfunctions $\beta_k(x) = \cos(kx)/\|\cos(kx)\|_{2,2}$. Hypotheses (H1)-(H4) hold for (4.20), hence the characteristic equation for (4.20) is decomposed into the sequence of characteristic equations

$$\lambda + ae^{-\lambda r} + dk^2 = 0 \quad (k \in \mathbb{N}_0), \tag{4.21$_k$}$$

and has been considered by many authors (see e.g. [15], [14, Chapter 3] and references therein).

As in (4.5), define $\gamma = ae$, and suppose that the delay r is small, so that

$$aer < 1.$$

Consider the above notations. If the diffusion rate is large enough, in the sense that $d > 1/r - ae$, we have $\Sigma_0 = \{\mu_0\} = \{0\}$. In this case, Eq. (4.7) reads as an FDE in $C_1 = C([-r,0];\mathbb{R})$,

$$\dot{x}(t) = -ax(t-r),$$

for which the special solution $u(t)$ satisfying $u(0) = 1$ is $u(t) = e^{\lambda_* t}$, where λ_* is the characteristic root of (4.21_0) in $(-1/r, 0)$, i.e., $\lambda_* + ae^{-\lambda_* r} = 0$ (see also Example 3.1). Consequently,

$$X(t) = e^{\lambda_* t}, \ t \in \mathbb{R},$$

$$F^0 = span\,\{\beta_0\}, \quad \beta_0 = \frac{1}{\sqrt{\pi}}, \quad P = span\,\{\varphi_0\}, \text{ where } \varphi_0(\theta) = e^{\lambda_* \theta}, -r \leq \theta \leq 0.$$

The formal duality in $C_1^* \times C_1$ is given by

$$< \psi, \phi >= \psi(0)\phi(0) - a\int_{-r}^{0} \psi(\xi + r)\phi(\xi)d\xi,$$

from which we get

$$Y^0(s) = Y(s) = ce^{-\lambda_* s}, \quad 0 \le s \le r, \quad \text{with} \quad c = (1 + r\lambda_*)^{-1}.$$

Let $C([-r, 0]; F^0)$ be identified with $C_1 = C([-r, 0]; \mathbb{R})$. The projection $\pi : C \to C([-r, 0]; F^0)$ is given by $(\pi\phi)(\theta) = (\phi(\theta), \beta_0)\beta_0 \equiv (\phi(\theta), \beta_0)$, where $(\cdot, \cdot)$ is the inner product in $\mathcal{X}$. Therefore, we obtain

$$< Y^0, \pi\phi >= (1 + r\lambda_*)^{-1}\Big[(\phi(0), \beta_0) - a\int_{-r}^{0} e^{-\lambda_*(\xi+r)}(\phi(\xi), \beta_0)\, d\xi\Big]. \tag{4.22}$$

If we further impose $3aer < 1$, i.e., if $\frac{1}{r} - d < ae < \frac{1}{3r}$, then (4.14) holds, and from Theorem 4.2 we deduce that

$$\lim_{t\to\infty} e^{-\lambda_* t} u(\phi)(t) =< Y^0, \pi\phi > \beta_0, \tag{4.23}$$

for $< Y^0, \pi\phi >$ given by (4.22). If $ae < \min\{\frac{1}{r}, d - \frac{1}{r}\}$, then condition (4.10) holds, and from Theorem 4.1 we deduce

$$u(\phi)(t) = e^{\lambda_* t} < Y^0, \pi\phi > \beta_0 + o(e^{-t/r}) \quad \text{as} \quad t \to \infty,$$

a condition stronger than (4.23).

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