On Non-Newtonian Incompressible Fluids with Phase Transitions

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Abstract

A modified model for a binary fluid is analyzed mathematically. The governing equations of the motion consists of a Cahn-Hilliard equation coupled with a system describing a class of non-Newtonian incompressible fluid with p-structure. The existence of weak solutions for the evolution problems is shown for the space dimension $d = 2$ with $p \geq 2$ and for $d = 3$ with $p \geq 11/5$. The existence of measure-valued solutions is obtained for $d = 3$ in the case $2 \leq p < 11/5$. Similar existence results are obtained for the case of nondifferentiable free energy, corresponding to the density constraint $|\psi| \leq 1$. We also give regularity and uniqueness results for the solutions and characterize stable stationary solutions.

1 Introduction

A two-phase flow is fluid motion which has two different phase states. When we consider a two-phase flow between immiscible fluids or a motion of sharp interfaces, it is necessary to take the effect of convection (fluidity) into account together with the free energy of the system. Dynamics of two-phase systems ignoring convection has been

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studied deeply in the literature and Cahn-Hilliard equation has been playing a central role in this area. Navier-Stokes equations also have been central in fluid mechanics. Thus, a coupling of Cahn-Hilliard and Navier-Stokes equations can be a first candidate to describe a phase transition phenomena with fluidity when the sharp interface is replaced by a narrow transition layer determined by both diffusion and motion. Indeed, there have been several papers introducing such models to describe fluctuations or hydrodynamic effects in the critical phenomena [8, 13, 18]. These models are variations of a model called model H in [9], which can be systematically derived [7] and generalized [15] or [5] (see also [1]). The model H is a system of incompressible Navier-Stokes equations coupled with Cahn-Hilliard equation through quadratic coupling terms (reversible modes). The model also encompasses the case of non-constant mobility and surface tension. In particular, for the static case, the model becomes similar to one in [6]. However, there is a difference between the two models since the model H admits a variational structure while the model in [6] does not. Phase field models have been also used with success for the numerical computations of interface movement using different methods, as for instance, in [10], [5] or [12]. Qualitative studies of the behavior of Cahn-Hilliard flow model were considered with Navier-Stokes equations ($p = 2$) and constant surface tension coefficient in [4] and also with constant mobility in [3], for slightly nonhomogeneous diphasic incompressible fluids under shear.

In this paper, we consider a convective phase field system for modified model H on a smooth bounded domain or on a torus for non-Newtonian fluids with p-structure. We first introduce the definitions and we prove the existence of weak solutions under relative density (order parameter) dependent viscosity, surface tension coefficient and mobility for $p \geq (3d+2)/(d+2)$ in the space dimensions, $d = 2$, 3, recovering the Ladyzhenskaya-Lions result [14]. The particular case of Navier-Stokes equations ($p = 2$) is also covered for $d = 3$. The Lyapunov functional turns out to fall into the classical case of the Cahn-Hilliard system for the static case. The Lyapunov functional actually guarantees the stability of local minimizers of the classical functional in the absence of external forces.

To fill the gap $2 \leq p < 11/5$ when $d = 3$, we prove the existence of measure-valued solutions for $p \geq 2$ ($d = 2, 3$) in the line of [16]. Then we show the uniqueness of weak solutions for $p \geq (d+2)/2$, $d = 2, 3$. Some regularity and existence results are obtained in two-dimensional space ($d = 2$) for a class of non-Newtonian fluids undergoing a well behaved stress tensor with p-growth, $p > 1$, when the viscosity, surface tension coefficient and mobility are constants. Finally in the last section, we consider the case of nondifferentiable free energy in order to obtain a solution satisfying the (physical) density constraint $|\psi| \leq 1$ as in [2] (see also [11]).

2 A convective phase field system

The state of the system is described by a pair (u, ψ), where $u = (u_1(x,t), \cdots, u_d(x,t))$ is the velocity field of the fluid and $\psi = \psi(x,t)$ is the order parameter (the relative density). The system of equations for (u, ψ) is

$$\partial_t u + (u \cdot \nabla) u = -\nabla q + \nabla \cdot (\nu\tau) - \nabla \cdot (\alpha \nabla\psi \otimes \nabla\psi) \tag{1}$$

$$\nabla \cdot u = 0 \tag{2}$$

$$\partial_t \psi + u \cdot \nabla\psi = \nabla \cdot (m\nabla(f'(\psi) - \sqrt{\alpha}\nabla \cdot (\sqrt{\alpha}\nabla\psi))) \tag{3}$$

with appropriate initial and boundary conditions. Here, $f \in C^2(\mathbb{R} \to \mathbb{R}^+)$ is a volumetric free energy, $\nu = \nu(\psi) > 0$ the viscosity, $\alpha = \alpha(\psi) > 0$ the surface tension coefficient, $m = m(\psi) > 0$ the mobility, $q = q(x,t)$ the scalar pressure, and $\tau = \tau(D(u))$ the viscous stress satisfying the $(p-1)$-growth and p-coercivity conditions, where $D(u) = D_{ij}(u) = (\partial_i u_j + \partial_j u_i)/2$ is the velocity of strain tensor, $i, j = 1, \cdots, d$. The system (1)-(3) is derived under the assumption of constant density and incompressibility. We assume that the coefficients ν, α, and m depend only on ψ. We also assume that ν, α, and m are continuous functions such that they are bounded from below and above by positive definite constants

$$0 < \gamma_1 \leq \nu(\psi), \alpha(\psi), m(\psi) \leq \gamma_2, \quad \forall \psi \in \mathbb{R}. \tag{4}$$

These assumptions are reasonable and used in the derivation of (1)-(3) in [7]. Due to physical motivation, we only consider f of double-well type satisfying the following conditions

$$\left.\begin{array}{l} f(y) \geq 0,\ f'(y)/f(y) = o(1) \ \text{ as } |y| \to \infty, \\ f(y) \ \text{ has local minima only at } \ y = \pm 1, \\ f(y) \ \text{ is strictly monotone for } \ |y| > 1 \end{array}\right\} \tag{5}$$

Denoting by $\mathbb{R}^{d^2}_{sym}$ the set of symmetric $d \times d$ matrices, a non-Newtonian fluid (see [16], for instance) can be described by a monotone $\tau_{ij} \in C(\mathbb{R}^{d^2}_{sym})$ such that

$$\tau_{ij}(0) = 0 \qquad |\tau(\zeta)| \leq \gamma_3(1 + |\zeta|)^{p-1} \tag{6}$$

$$\tau(\zeta) : \zeta \geq \gamma_4 |\zeta|^p, \quad \forall \zeta \in \mathbb{R}^{d^2}_{sym}. \tag{7}$$

We shall study the initial boundary value problem of the above system on two types of domains, Ω_B and Ω_P. $\Omega_B \subset \mathbb{R}^d$ is a smooth bounded domain and Ω_P is the usual d-torus. For these domains, we work with different boundary conditions

$$u = \frac{\partial}{\partial n}\psi = \frac{\partial}{\partial n}\mu = 0 \quad \text{on } \partial\Omega_B \tag{8}$$

or

$$u(x) = u(x+\mathbf{e}), \psi(x) = \psi(x+\mathbf{e}), \mu(x) = \mu(x+\mathbf{e}) \quad \text{on } \partial\Omega_P. \tag{9}$$

Here, $\mathbf{e}$ is an element of a basis of the torus, n is the outward normal vector of Ω_B, and

$$\mu = f'(\psi) - \sqrt{\alpha}\nabla\cdot(\sqrt{\alpha}\nabla\psi). \tag{10}$$

μ is called the chemical potential in the physical literature. The boundary condition (8) is physically more meaningful than the Dirichlet type condition. In fact, we can work with the Dirichlet type boundary condition, $u = \psi = \Delta\psi = 0$ on $\partial\Omega_B$ instead of (8) to get similar results. Those boundary conditions can be incorporated in the functional spaces below. First we define the following spaces

$$W_B^{k,2} = \{\psi \in W^{k,2} | \frac{\partial^{k-1}\psi}{\partial^{k-1}n} = 0 \ \ on \ \ \partial\Omega_B\}, \quad k \geq 2$$

$$\mathcal{V}_B(\mathcal{V}_P) = \{u | u \in (C_0^\infty)^d((C_{per}^\infty)^d), \nabla\cdot u = 0\},$$

$$\mathbf{J}_B^{k,p}(\mathbf{J}_P^{k,p}) = \{u | u \in (W_0^{k,p})^d((W_{per}^{k,p})^d), \nabla\cdot u = 0\}, \quad k \geq 0, \ p > 1.$$

Then the spaces we work with will be

$$\begin{aligned} \mathbf{H}_B(\mathbf{H}_P) &= \mathbf{J}_B^{0,2} \times W^{1,2}(\mathbf{J}_P^{0,2} \times W_{per}^{1,2}) \\ \mathbf{V}_B^p(\mathbf{V}_P^p) &= \mathbf{J}_B^{1,p} \times W_B^{3,2}(\mathbf{J}_P^{1,p} \times W_{per}^{3,2}). \end{aligned}$$

We denote by Ω either of the two domains Ω_B and Ω_P, and we understand $\mathbf{H} = \mathbf{H}_B(\mathbf{H}_P)$, $\mathbf{J}^{k,p} = \mathbf{J}_B^{k,p}(\mathbf{J}_P^{k,p})$, $W^{k,p} = W_B^{k,p}(W_{per}^{k,p})$, and $\mathbf{V}^p = \mathbf{V}_B^p(\mathbf{V}_P^p)$ respectively. Clearly, $\mathbf{H}$ and $\mathbf{V}^p$ are Banach spaces and $\mathbf{V}^p$ is compactly embedded in $\mathbf{H}$ for $d = 2, 3$, $p \geq 2$. We also use the summation convention throughout this paper.

We first define a weak solution of (1)-(3) and afterwards we extend the definition to allow measure-valued solutions.

Definition 1 *We say that (u, ψ, μ) is a weak solution of (1)-(3) on Ω with initial data (u_0, ψ_0) for $0 < t < T$ if $(u, \psi) \in L^\infty(0, T; \mathbf{H})$, $u \in$*

$L^p(0,T;\mathbf{J}^{1,p})$, $\nabla\mu \in L^2(0,T;\mathbf{L}^2)$, $f(\psi) \in L^1(0,T;L^1)$, *and for any test function* $(v,\phi) \in \mathbf{V}^p$, (u,ψ,μ) *satisfies the following formulation*

$$\begin{aligned}\int_\Omega v\cdot u(t) &- \int_\Omega v\cdot u_0 \\ &= \int_0^t \int_\Omega (u_i u_j \partial_i v_j - \nu\tau_{ij}(Du)\partial_i v_j - \psi v\cdot\nabla\mu) \quad (11)\\ \int_\Omega \phi\psi(t) &- \int_\Omega \phi\psi_0 = \int_0^t\int_\Omega (\psi u\cdot\nabla\phi - m\nabla\phi\cdot\nabla\mu) \quad (12)\\ \int_\Omega \mu\phi &= \int_\Omega (f'(\psi)\phi + \sqrt{\alpha}\nabla\psi\cdot\nabla(\sqrt{\alpha}\phi)) \quad a.e.\ t. \quad (13)\end{aligned}$$

Note that all terms in (11) - (13) are meaningful. Indeed, $W^{1,p} \hookrightarrow L^4$ for $p \geq 3d/(d+2)$ and $d \leq 4$, $\partial_t u \in L^1(0,T;(\mathbf{J}^{1,p})')$ for $p \geq (d+\sqrt{3d^2+4d})/(d+2)$, and $\partial_t u \in L^{p'}(0,T;(\mathbf{J}^{1,p})')$ for $p \geq (3d+2)/(d+2)$. The above equations are formally equivalent to (1)-(3) since

$$\begin{aligned}\nabla\cdot(\alpha\nabla\psi\otimes\nabla\psi) &= \sqrt{\alpha}\nabla\cdot(\sqrt{\alpha}\nabla\psi)\nabla\psi + \frac{1}{2}\nabla(\alpha|\nabla\psi|^2)\\ &= -\mu\nabla\psi + \nabla(\frac{1}{2}\alpha|\nabla\psi|^2 + f(\psi)).\end{aligned}$$

In order to define a measure-valued solution as in [16], we recall the space of probability measures:

$$\mathrm{Prob}(\mathbb{R}^s) \equiv \{\lambda \in \mathcal{M}(\mathbb{R}^s),\ \lambda \geq 0\ \lambda(\mathbb{R}^s) = 1\}$$

where $\mathcal{M}(\mathbb{R}^s)$ denotes the space of bounded Radon measures on $\mathbb{R}^s$. By $\lambda \in L^\infty_w(\Omega\times(0,T);\mathcal{M}(\mathbb{R}^s))$, we mean that $\lambda : \Omega\times(0,T) \to \mathcal{M}(\mathbb{R}^s)$ is a weak measurable function, that is, the function

$$\begin{aligned}(x,t) &\mapsto \langle\lambda_{x,t}, F((x,t),\cdot)\rangle\\ &= \int_{\mathbb{R}^s} F((x,t);\eta)d\lambda_{x,t}(\eta) \qquad \forall F \in L^1(\Omega\times(0,T);C_0(\mathbb{R}^s))\end{aligned}$$

is measurable. Moreover, the norm

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega\times(0,T)} \|\lambda_{x,t}\|_{\mathcal{M}(\mathbb{R}^s)}$$

is finite.

Definition 2 *We say that* (u,λ,ψ,μ) *is a measure-valued solution of (1)-(3) on* Ω *with initial data* (u_0,ψ_0) *for* $0 < t < T$ *if* $(u,\psi) \in L^\infty(0,T;\mathbf{H})$, $\nabla u \in L^p(0,T;\mathbf{L}^p)$, $\lambda \in L^\infty_w(\Omega\times(0,T);\mathrm{Prob}(\mathbb{R}^{d^2}))$, $\nabla\mu \in$

$L^2(0,T;\mathbf{L}^2)$, $f(\psi) \in L^1(0,T;L^1)$, and (u,λ,ψ,μ) satisfies (12) -(13) and the following formulation

$$\int_\Omega v \cdot u(t) - \int_\Omega v \cdot u_0 = \int_0^T \int_\Omega \Big(u_i u_j \partial_i v_j \\ -\nu \partial_i v_j \int_{\mathbb{R}^{d^2}} \tau_{ij} \left(\frac{\eta + \eta^T}{2} \right) d\lambda_{x,t}(\eta) - \psi v \cdot \nabla \mu \Big) \tag{14}$$

for all $v \in \mathcal{V}$ and $\phi \in W^{3,2}_B(W^{3,2}_{\text{per}})$; and

$$\partial_j u_i(x,t) = \int_{\mathbb{R}^{d^2}} \eta_{ij} d\lambda_{x,t}(\eta) \qquad a.e.\ in\ \Omega \times (0,T). \tag{15}$$

Remark 1 *When the measure $\lambda_{x,t} = \left(\delta_{\partial_j u_i(x,t)}\right)$ is a Dirac measure at almost every point $(x,t) \in \Omega \times (0,T)$ we have*

$$\tau_{ij}(D(u)) = \int_{\mathbb{R}^{d^2}} \tau_{ij} \left(\frac{\eta + \eta^T}{2} \right) d\lambda_{x,t}(\eta)$$

and a weak solution is also a special measure-valued solution.

3 Existence and Lyapunov functional

First, we recall a special case of the Gargliardo-Nirenberg inequality [17] which will be used in this paper.

Lemma 1 *Let i, j, and k be non-negative integers, $j \le i < k$ and either $v \in H^k_0(\Omega_B)$ or $v \in H^k_{per}(\Omega_P)$ with $\int_{\Omega_P} \nabla^i v = 0$. Then*

$$\|\nabla^i v\|_{L^p} \le C \|\nabla^j v\|^a_{L^2} \|\nabla^k v\|^{1-a}_{L^2}, \quad a = \frac{(2k-2i-d)p + 2d}{2(k-j)p}, \tag{16}$$

$$\begin{array}{ll} 2 \le p \le 2d/(2i+d-2k) & \text{if } \ 2i+d > 2k \\ 2 \le p \le \infty & \text{if } \ 2i+d < 2k. \end{array}$$

Let us prove some useful a priori estimates.

Lemma 2 *Given (u,ψ), a smooth solution of (1)-(3) with (8) or (9), we have*

$$\int_\Omega \psi(t) = \int_\Omega \psi(0), \tag{17}$$

$$Q(t) + 2\int_0^t \int_\Omega \left[\nu \tau_{ij} \partial_i u_j + m |\nabla \mu|^2 \right] \le Q(0) \tag{18}$$

where
$$Q = \int_\Omega (u^2 + 2f(\psi) + \alpha|\nabla\psi|^2)$$
and μ as in (10). As a consequence, for a constant $C > 0$
$$Q(t) + C\int_0^t \int_\Omega \left[|\nabla u|^p + |\nabla\mu|^2\right] \le Q(0). \qquad (19)$$

Proof) First, (17) can be obtained easily. Indeed, integrating (3) and using the divergence theorem, we recover (17) since the boundary terms vanish due to the boundary conditions. Next, we multiply (1) by u and (3) by μ, then add them after integrating them. Using the divergence theorem, we have
$$\int_\Omega u\cdot\nabla u_i u_i = 0,$$
$$\int_\Omega \nabla\cdot(\nu\tau)\cdot u = -\int_\Omega \nu\tau_{ij}\cdot\partial_i u_j,$$
$$\int_\Omega \partial_t(2f(\psi) + |\sqrt{\alpha}\nabla\psi|^2) = 2\int_\Omega \mu\partial_t\psi,$$
$$\begin{aligned}\int_\Omega (u\cdot\nabla\psi)\mu &= \int_\Omega \nabla\cdot(f(\psi)u) - \int_\Omega u\cdot\sqrt{\alpha}\nabla\psi\nabla\cdot(\sqrt{\alpha}\nabla\psi)\\ &= -\int_\Omega u\cdot\nabla\cdot(\alpha\nabla\psi\otimes\nabla\psi).\end{aligned}$$
Using the above identities and integrating with respect to the time, we arrive at (18). Since $\tau_{ij}\partial_i u_j = \tau_{ij}D_{ij}(u) \ge 0$,
$$\int_\Omega \nu\tau_{ij}\partial_i u_j \ge \int_\Omega \gamma_1\tau_{ij}\partial_i u_j \ge \gamma_1\gamma_4\int_\Omega |\nabla u|^p$$
by the assumption (7). Using this fact, we reduce (18) to (19). ∎

In view of Lemma 2, we shall denote from now on
$$M = \int_\Omega \psi_0.$$

We shall analyze separately the Navier-Stokes case and the case $p \ge 11/5$ in three dimensional space.

Theorem 1 *Given an initial data $(u_0, \psi_0) \in \mathbf{H}$ with $f(\psi_0) \in L^1$, for $p \ge (3d+2)/(d+2)$ there exists a weak solution $(u,\psi) \in L^\infty(0,T;\mathbf{H})$, $u \in L^p(0,T;\mathbf{J}^{1,p})$, $\mu \in L^2(0,T;W^{1,2})$ for any $T > 0$.*

Proof) We use the Faedo-Galerkin argument. We first show the theorem for f growing at most quadratically near infinity. Let $\{\xi_i, i \in \mathbb{N}\}$ and $\{\rho_i, i \in \mathbb{N}\}$ be a smooth orthonormal basis of $\mathbf{J}^{0,2}$ and $W^{1,2}(W^{1,2}_{per})$ respectively. Clearly, $\rho_1 = 1/|\Omega|$ and $(\xi_i, \rho_j),\ i,j \in \mathbb{N}\}$ forms an orthonormal basis for $\mathbf{H}$. And, let P_1^i and P_2^i, $i \in \mathbb{N}$ be the projection operators onto $span(\xi_1, \cdots, \xi_i)$ and $span(\rho_1, \cdots, \rho_i)$ respectively. We consider the approximate solutions, $(u^i, \psi^i, \mu^i) \in span((\xi_1, \rho_1, \rho_1), \cdots, (\xi_i, \rho_i, \rho_i)),\ i \in \mathbb{N}$ of the following system

$$\partial_t u^i + P_1^i(u^i \cdot \nabla u^i) = P_1^i(\nabla \cdot (\nu^i \tau^i)) + P_1^i(\mu^i \nabla \psi^i) \tag{20}$$

$$\partial_t \psi^i + P_2^i(u^i \cdot \nabla)\psi^i = P_2^i \nabla \cdot (m^i \nabla \mu^i) \tag{21}$$

$$\mu^i = P_2^i(f'(\psi^i) - \sqrt{\alpha^i}\nabla \cdot (\sqrt{\alpha^i}\nabla\psi^i)), \tag{22}$$

where α^i, ν^i, m^i, and τ^i_{jk} correspond to (u^i, ψ^i). We note that P_2^i in (22) makes the system consistent and is useful to obtain the essential estimates. For any $i \in \mathbb{N}$, the above system is a system of ODEs thus, for the initial data $(u_0^i, \psi_0^i) \equiv (P_1^i u_0, P_2^i \psi_0)$, the above system has a (local in time) unique solution, (u^i, ψ^i, μ^i). Exactly as in Lemma 2 using the idempotency of the projection operators, (u^i, ψ^i, μ^i) satisfies (18) and $P_2^1 \psi^i(t) = P_2^1 \psi_0,\ i \in \mathbb{N}$ like (17). Since f grows at most quadratically in this case,

$$\int_\Omega f(\psi_0^i) \le C + C\int_\Omega |\psi_0^i|^2 \le C + C\int_\Omega |\psi_0|^2.$$

Then, $Q(u_0^i, \psi_0^i) \le CQ(u_0, \psi_0)$ and thus $Q(u^i, \psi^i)(t) \le CQ(u_0, \psi_0)$ by (19). By (22),

$$\begin{aligned} |\int_\Omega \mu^i| &= |\Omega|| \int_\Omega \mu^i \cdot \rho_1| = |\int_\Omega (f'(\psi^i) + \sqrt{\alpha^i}\nabla\psi^i \cdot \nabla\sqrt{\alpha^i})| \\ &\le C + C\int_\Omega f(\psi^i) + C\int_\Omega |\nabla\psi^i|^2 \le C + CQ(u_0, \psi_0). \end{aligned}$$

Thus, by (19), the above inequality, and the Poincaré inequality, $\mu^i \in W^{1,2}$ uniformly. Due to the continuity of the local solution (u^i, ψ^i, μ^i), $i \in \mathbb{N}$ and its uniform boundedness in time, we can shift to time T and repeat the above process. Then, for any $T > 0$, $(u^i, \psi^i) \in L^\infty(0,T;\mathbf{H})$, $\nabla u^i \in L^p(0,T;\mathbf{L}^p)$, and $\mu^i \in L^2(0,T;W^{1,2})$ uniformly with respect to $i \in N$.

Next, we multiply (20) and (21) by $v \in \mathbf{J}^{1,p}$ and $\phi \in W^{1,2}(W^{1,2}_{per})$ respectively to calculate $\|\partial_t u^i\|_{(\mathbf{J}^{1,p})'}$ and $\|\partial_t \psi^i\|_{H^{-1}}$. Indeed,

$$\begin{aligned} |\langle \partial_t u^i, v\rangle| &= |\int_\Omega u^i \otimes u^i \cdot \nabla P_1^i v - \int_\Omega \nu^i \tau^i \cdot \nabla P_1^i v - \int_\Omega \psi^i P_1^i v \cdot \nabla \mu^i| \\ &\le C(\|u^i\|^2_{L^{2p/(p-1)}} + \gamma_2\gamma_3(1 + \|\nabla u^i\|^{p-1}_{L^p}))\|P_1^i v\|_{W^{1,p}} \\ &\quad + \|\psi^i\|_{L^4}\|P_1^i v\|_{L^4}\|\nabla\mu^i\|_{L^2}; \end{aligned}$$

here, we used the fact, $\nabla \cdot P_1^i v = 0$ and $p \geq 3d/(d+2)$. Subsequently,

$$|\langle \partial_t u^i, v\rangle| \leq C(\|u^i\|_{L^2}^{2(1-\beta)} \|\nabla u^i\|_{L^p}^{2\beta} + \gamma_2\gamma_3(1 + \|\nabla u^i\|_{L^p}^{p-1}) + \\ + \|\psi^i\|_{H^1}\|\nabla \mu^i\|_{L^2})\|P_1^i v\|_{W^{1,p}}$$

by the interpolation and Sobolev inequalities for $\beta = d/[(d+2)p - 2d]$. With the fact $\|P_1^i v\|_{W^{1,p}} \leq \|v\|_{\mathbf{J}^{1,p}}$, (17), the Poincaré inequality, and (19), and choosing $\delta \geq 1$ such that $2\beta\delta \leq p$ and $\delta \leq p'$, we deduce

$$\int_0^T \|\partial_t u^i\|_{(\mathbf{J}^{1,p})'}^{\delta} \leq C(T + M^4 + Q(u_0,\psi_0)^{2\delta}) \tag{23}$$

for any $T > 0$. The limit case $\delta = 1$ corresponds to the values $p \geq (d + \sqrt{3d^2 + 4d})/(d+2)$ already found in [16, pp. 220].

Considering

$$\begin{aligned} |\int_\Omega \partial_t \psi^i \phi| &= |\int_\Omega \psi^i (u^i \cdot \nabla) P_2^i \phi - \int_\Omega \nabla \cdot (P_2^i \phi) m^i \nabla \mu^i| \\ &\leq (\|\psi^i\|_{L^4}\|u^i\|_{L^4} + \gamma_2 \|\nabla \mu^i\|_{L^2})\|\nabla P_2^i \phi\|_{L^2}, \end{aligned}$$

and applying the Sobolev and Poincaré inequalities, (17), and (19), we have

$$\int_0^T \|\partial_t \psi^i\|_{H^{-1}}^2 \leq C(1 + M^4 + Q(u_0,\psi_0)^2) \tag{24}$$

for any $T > 0$. Therefore, for any $T > 0$, using a well-known compactness theorem [14], we can find a subsequence of u^i converging strongly in $L^p(0,T;\mathbf{J}^{0,2})$, since $W^{1,p} \hookrightarrow\hookrightarrow L^2$ if $p > 2d/(d+2)$, a subsequence of ψ^i converging strongly in $L^2(0,T;L^2)$, and a subsequence of μ^i converging weakly in $L^2(0,T;W^{1,2})$. We denote the limits by u, ψ, and μ respectively. Then, for any $T > 0$, $u \in L^\infty(0,T;\mathbf{J}^{0,2}) \cap L^p(0,T;\mathbf{J}^{1,p})$, $\psi \in L^\infty(0,T;W^{1,2})$, and Lemma 2 holds for (u,ψ,μ).

To pass to the limit the nonlinear term, we refer that the density dependent coefficient keeps the monotonicity property as

$$\int_\Omega (\nu(\psi^i)\tau^i - \nu(\psi^j)\tau^j) : (D^i - D^j) = \int_\Omega \nu(\psi^i)(\tau^i - \tau^j) : (D^i - D^j) \\ + \int_\Omega [\nu(\psi^i) - \nu(\psi^j)]\tau^j : (D^i - D^j),$$

for two solutions (u^i, ψ^i, μ^i) and (u^j, ψ^j, μ^j). Thus applying monotone arguments (see [14]) where the convective term has meaning if and only if $p \geq (3d+2)/(d+2)$ which corresponds to $\delta = p'$, the limits satisfy (11) and trivially (12). Since $f'/f = o(1)$ and $f(\psi) \in L^\infty(0,T;L^1)$, (13) also holds for μ.

We next consider the case of f growing faster. In this case, we can approximate f by a sequence of $f_j \geq 0$, $j \in \mathbb{N}$ growing at most quadratically and satisfying $f_1 \leq f_2 \leq \cdots \leq f$. In fact, we can define that $f_j(y) = f(y)$ for $|y| < j$, $f_j(y) = 1/2(f(j) + f(j+1))$ for $|y| > j+1$, and then make a smooth and monotone interpolation in between. Then, we have a sequence of solutions (u^j, ψ^j, μ^j) for each f_j which satisfies all the above results. For each f_j, $j \in \mathbb{N}$, $Q(u_0, \psi_0)(f_j) \leq Q(u_0, \psi_0)(f)$ since $f_j \leq f$. Thus (u^j, ψ^j, μ^j) is again a bounded sequence and we can find a limit (up to a subsequence) (u, ψ, μ) under the same topology as before. By Fatou's lemma, we further deduce (u, ψ, μ) satisfy (19). The limit is verified to satisfy (11) and (12) in a similar fashion as before. Using the fact $f(\psi) \in L^1$ and (5), we can also show (13). □

The corollary of the above theorem shows that the space of the order parameter solution is actually similar to that of the classical Cahn-Hilliard equation.

Corollary 1 *Under all the assumptions of the above theorem, suppose further that α is a constant and that*

$$|f''(y)| \leq C(1 + |y|^r), \quad \left.\begin{array}{ll} r = 3 & \text{if } d = 3 \\ \text{for any } r > 0 & \text{if } d = 2. \end{array}\right\} \tag{25}$$

Then, the following estimate holds

$$\int_0^T \|\nabla^3 \psi\|_{L^2}^2 \leq C(1 + T + M^{8r/(4-d)} + Q(0)^{4r/(4-d)})Q(0) \tag{26}$$

for any $p > 1$.

Proof) Considering that all components of the Galerkin system ρ_i, $i \in \mathbb{N}$ are eigenvectors of $-\Delta$, we only need to show that $\Delta\nabla\psi \in L^2(0, T; \mathbf{L}^2)$. By (13), $\alpha\Delta\nabla\psi = f''(\psi)\nabla\psi - \nabla\mu$. Using (25) and (16) it follows

$$\begin{aligned} \|f''(\psi)\nabla\psi\|_{L^2} &\leq \|f''(\psi)\|_{L^2}\|\nabla\psi\|_{L^\infty} \\ &\leq C(1 + \|\psi\|_{L^{2r}}^r)\|\nabla\psi\|_{L^2}^{1-d/4}\|\nabla\Delta\psi\|_{L^2}^{d/4}. \end{aligned}$$

Applying the Poincaré inequality, we have

$$\|\psi\|_{L^{2r}} \leq C(M + \|\nabla\psi\|_{L^2})$$

for both of the cases $r = 3$, $d = 3$ and any $r > 0$, $d = 2$. Then we infer

$$\|\Delta\nabla\psi\|_{L^2} \leq C\|\nabla\mu\|_{L^2} + C(1 + M^{4r/(4-d)} + \|\nabla\psi\|_{L^2}^{4r/(4-d)})\|\nabla\psi\|_{L^2}$$

using the Young inequality. This proves (26) taking into account (18).

∎

Corollary 2 *Under all the assumptions of the above corollary, the existence of weak solution remains valid in the case of Newtonian fluids for* $\tau = D(u)$ *with* $p = 2$ *and* $d = 3$*, and we have further* $(u, \psi) \in L^2(0, T; \mathbf{V}^2)$.

Proof) The proof for the case $\tau = D(u)$ and $d = 3$ is identical to the proof of Theorem 1, since we can derive the same estimates for the Galerkin approximations and the weak convergence of ∇u^i to ∇u in $\mathbf{L}^2$ is sufficient to pass to the limit (20). Then, we have further $(u, \psi) \in L^2(0, T; \mathbf{V}^2)$, taking into account the regularity property given at the above corollary. ∎

Corollary 3 *In the absence of the external forces,* $(u, \psi) \in \mathbf{H}$ *is a stationary stable solution if and only if* $u = 0$ *and* ψ *is a local minimizer of*

$$Q_{cl}(\psi) = \int_\Omega \alpha |\nabla \psi|^2 + 2f(\psi).$$

Proof) As Q is a Lyapunov functional of the system (1)-(3), $(0, \psi)$ is a stable stationary solution in $\mathbf{H}$ if ψ is a local minimizer of Q_{cl}. On the contrary, if (u_1, ψ_1) is a stable stationary solution in $\mathbf{H}$, we can consider the Cauchy problem with initial data (u_1, ψ_1). Then the solution obtained by Theorem 1 must satisfy (18). However, the solution is just (u_1, ψ_1), which means $u_1 = 0$ and therefore $Q = Q_{cl}$. Since (u_1, ψ_1) is stable, ψ_1 is a local minimizer of Q_{cl}. ∎

4 Measure-valued solutions

Let us recall first the following consequence of a theorem on Young measures which is the basis to the existence result of measure-valued solutions (cf. Corollary 2.10, [16, pp. 172]).

Lemma 3 *Let* $Q \subset \mathbb{R}^d$ *be a bounded open set. Let* z^i *be uniformly bounded in* $L^p(Q)^s$*. Then there exists a subsequence still denoted by*

z^i and a measure-valued function λ, such that, for all $\tau : \mathbb{R}^s \to \mathbb{R}$ satisfying for some $q > 0$ the growth condition

$$|\tau(\eta)| \leq C(1 + |\eta|)^q \quad \forall \eta \in \mathbb{R}^s$$

we have

$$\tau(z^i) \rightharpoonup \bar{\tau} \qquad \text{in } L^r(Q)$$

where

$$\bar{\tau}(y) = \langle \lambda_y, \tau \rangle \qquad \text{a.e. in } Q$$

provided that $1 < r \leq p/q$.

Theorem 2 *Given an initial data $(u_0, \psi_0) \in \mathbf{H}$ with $f(\psi_0) \in L^1$, for $p \geq 2, (d = 2, 3)$, there exists a measure-valued solution $(u, \psi) \in L^\infty(0, T; \mathbf{H})$ for any $T > 0$.*

Proof) As in the proof of weak solutions, we first start with the case f growing at most quadratically near infinity. Let $\{\xi_i,\ i \in \mathbb{N}\}$ and $\{\rho_i,\ i \in \mathbb{N}\}$ be an orthonormal basis of $\mathbf{J}^{k,2}$, $k > 1 + d/2$ (cf. [16, pp. 206]), and $W^{1,2}(W^{1,2}_{per})$ respectively. Then there exists an approximate solution (u^i, ψ^i, μ^i), $i \in \mathbb{N}$ such that, for any $T > 0$, $(u^i, \psi^i) \in L^\infty(0, T; \mathbf{H})$, $\nabla u^i \in L^p(0, T; \mathbf{L}^p)$, $\partial_t \psi^i \in L^2(0, T; H^{-1})$, $\nabla \mu^i \in L^2(0, T; \mathbf{L}^2)$, and $\mu^i \in L^2(0, T; W^{1,2})$ uniformly with respect to $i \in \mathbb{N}$.

However the estimate (23) is not valid for $p < 3d/(d+2)$, that is, $2 \leq p < 11/5$ when $d = 3$. In order to prove an estimate for $\partial_t u^i$ in $L^{p'}(0, T; (\mathbf{J}^{k,2})')$, we take $v \in L^p(0, T; \mathbf{J}^{k,2})$ such that $\|v\|_{L^p(0,T;\mathbf{J}^{k,2})} \leq 1$ in (20). It follows

$$\begin{aligned}
|\int_0^T \langle \partial_t u^i, v \rangle dt| &\leq C \int_0^T \|u^i\|^2_{L^2} \|\nabla P^i_1 v\|_{L^\infty} + (1 + \|\nabla u^i\|^{p-1}_{L^p}) \|\nabla P^i_1 v\|_{L^p} \\
&\quad + \|\psi^i\|_{L^2} \|P^i_1 v\|_{L^\infty} \|\nabla \mu^i\|_{L^2}\ dt \\
&\leq C(\|u^i\|^2_{L^\infty(0,T;L^2)} + 1 + \|\nabla u^i\|^{p-1}_{L^p(0,T;L^p)} + \\
&\quad + \|\psi^i\|_{L^\infty(0,T;L^2)} \|\nabla \mu^i\|_{L^2(0,T;L^2)}) \|P^i_1 v\|_{L^p(0,T;W^{k,2})}.
\end{aligned}$$

Remarking that $\|P^i_1 v\|_{W^{k,2}} \leq \|v\|_{\mathbf{J}^{k,2}}$ and that $k > 1 + d/2$, we have $\nabla v \in (W^{k-1,2})^{d \times d} \hookrightarrow (L^\infty)^{d \times d}$. Then the apriori estimate holds and the limit processes follow as in the proof of Theorem 1 except for the term

$$\int_0^T \int_\Omega \nu(\psi^i) \tau^i :\ D(v)$$

for all $v \in \mathcal{V}$. Applying Lemma 3 with $Q = \Omega \times (0, T)$, $z^i = D(u^i)$, $q = p - 1$, $r = p'$ and $s = d^2$, we have

$$\tau^i = \tau(D(u^i)) \rightharpoonup \bar{\tau} \quad \text{ in } L^{p'}(\Omega \times (0, T))^{d^2}$$

where

$$\bar{\tau}_{ij}(x,t) = \int_{\mathbb{R}^{d^2}} \tau_{ij}\left(\frac{\eta+\eta^T}{2}\right) d\lambda_{x,t}(\eta) \qquad \text{a.e. in } \Omega\times(0,T).$$

Therefore, since $\psi^i \to \psi$ a.e. in $\Omega\times(0,T)$ and ν is a continuous function satisfying (4), we conclude

$$\int_0^T\int_\Omega \nu(\psi^i)\tau^i :\ D(v) \longrightarrow \int_0^T\int_\Omega \nu(\psi)\bar{\tau} :\ D(v)$$

for all $v \in \mathcal{V}$.

The assertion (15) is obtained as in [16, pp. 212], that is, applying Lemma 3 with $\tau=$id, $q=1$, $r=p$ and $s=d^2$. $\square$

5 Uniqueness of weak solutions

In this section, we assume that

$$\alpha,\ m,\ \nu \text{ are positive constants},$$

and the viscous stress tensor τ satisfies (6) and, for some constant $\gamma_5 > 0$,

$$(\tau(\zeta)-\tau(\xi)) : (\zeta-\xi) \geq \gamma_5|\zeta-\xi|^p, \quad \forall \zeta,\xi \in \mathbb{R}^{d^2}_{sym} \tag{27}$$

under the restriction $p \geq 2$ (cf. [16, pp. 198]). Let us prove uniqueness for this case when $p \geq (d+2)/2$.

Theorem 3 *Assume $p \geq (d+2)/2$ and (25). Then, there exists a unique weak solution for (1)-(3) for a given initial data, $(u_0,\psi_0) \in \mathbf{H}$ with $f(\psi_0) \in L^1$.*

Proof) Let (v_1,ψ_1), (v_2,ψ_2) be two weak solutions given by Theorem 1 for the same initial data and let $(\overline{v},\overline{\psi}) = (v_1-v_2,\psi_1-\psi_2)$. Since we can take $\phi=1$ in (12), we can assume $\int_\Omega \psi_k(t) = \int_\Omega \psi_0$, $k=1,2$. In particular, $\int_\Omega \overline{\psi} = 0$. Similarly, $\int_\Omega \overline{v} = 0$ when $\Omega = \Omega_P$. This fact allows the application of (16) for $i=j=0$ in several occasions. We subtract the equations for (v_2,ψ_2) from (v_1,ψ_1) and integrate them after multiplying by $(\overline{v},\overline{\psi})$ to get

$$\begin{aligned}
\partial_t \int_\Omega |\overline{\psi}|^2 + 2m\alpha\int_\Omega |\Delta\overline{\psi}|^2 &\leq 2m\int_\Omega |\Delta\overline{\psi}||f'(\psi_1)-f'(\psi_2)| \\
&\quad +2\int_\Omega |\overline{v}\cdot\nabla\psi_2\overline{\psi}| \\
&\leq C\|\Delta\overline{\psi}\|_{L^2}\|f'(\psi_1)-f'(\psi_2)\|_{L^2} \\
&\quad +C\|\overline{v}\|_{L^2}\|\nabla\psi_2\|_{L^\infty}\|\overline{\psi}\|_{L^2},
\end{aligned}$$

$$
\begin{aligned}
\partial_t \int_\Omega |\overline{v}|^2 + 2\nu\gamma_5 \int_\Omega |\nabla \overline{v}|^2 \quad &\le \quad 2\alpha \int_\Omega (|\nabla\Delta\psi_1||\overline{v}\overline{\psi}| + |\Delta\overline{\psi}||\nabla\psi_2||\overline{v}|) \\
&\quad +2\int_\Omega |\overline{v}\otimes\overline{v} : \nabla v_2| \\
&\le \quad C\|\nabla\Delta\psi_1\|_{L^2}\|\overline{v}\overline{\psi}\|_{L^2} + \epsilon\int_\Omega |\Delta\overline{\psi}|^2 \\
&\quad +C_\epsilon\|\overline{v}\|^2_{L^2}\|\nabla\psi_2\|^2_{L^\infty} \\
&\quad +C\|\nabla v_2\|_{L^p}\|\overline{v}\|^2_{L^{2p/(p-1)}};
\end{aligned}
$$

here, $\epsilon > 0$ is arbitrary. Using the mean value theorem and (16), we have

$$
\|f'(\psi_1) - f'(\psi_2)\|_{L^2} \le \|\overline{\psi}\|_{L^4}\|f''(\xi)\|_{L^4} \le C\|\overline{\psi}\|^{3/4}_{L^2}\|\Delta\overline{\psi}\|^{1/4}_{L^2}\|f''(\xi)\|_{L^4}
$$

for some measurable $\xi(x) \in [\psi_1(x), \psi_2(x)]$ a.e. $x \in \Omega$. While,

$$
\begin{aligned}
\|\nabla\Delta\psi_1\|_{L^2}\|\overline{v}\overline{\psi}\|_{L^2} \quad &\le \quad \|\nabla\Delta\psi_1\|_{L^2}\|\overline{v}\|_{L^4}\|\overline{\psi}\|_{L^4} \\
&\le \quad C\|\nabla\Delta\psi_1\|_{L^2}\|\overline{v}\|^{1/2}_{L^2}\|\nabla\overline{v}\|^{1/2}_{L^2}\|\overline{\psi}\|^{3/4}_{L^2}\|\Delta\overline{\psi}\|^{1/4}_{L^2} \\
&\le \quad C_\epsilon\|\nabla\Delta\psi_1\|_{L^2}\|\overline{v}\|^2_{L^2} + C_\epsilon\|\nabla\Delta\psi_1\|^2_{L^2}\|\overline{\psi}\|^2_{L^2} \\
&\quad +\epsilon\|\Delta\overline{\psi}\|^2_{L^2} + \epsilon\|\nabla\overline{v}\|^2_{L^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\nabla v_2\|_{L^p}\|\overline{v}\|^2_{L^{2p/(p-1)}} \quad &\le \quad \|\nabla v_2\|_{L^p}\|\overline{v}\|^{(2p-d)/p}_{L^2}\|\nabla\overline{v}\|^{d/p}_{L^2} \\
&\le \quad C_\epsilon\|\nabla v_2\|^{2p/(2p-d)}_{L^2}\|\overline{v}\|^2_{L^2} + \epsilon\|\nabla\overline{v}\|^2_{L^2}.
\end{aligned}
$$

Then, taking ϵ small enough and denoting $A = \|\overline{v}\|^2_{L^2} + \|\overline{\psi}\|^2_{L^2}$, we obtain

$$
\partial_t A \le CA(1 + \|\nabla\psi_2\|^2_{L^\infty} + \|f''(\xi)\|^{8/3}_{L^4} + \|\nabla\Delta\psi_1\|^2_{L^2} + \|\nabla v_2\|^{2p/(2p-d)}_{L^2}).
$$

Due to the assumption on f, we have

$$
\begin{aligned}
\|f''(\xi)\|_{L^4} \quad &\le \quad C(1 + \||\xi|^r\|_{L^4}) \le C(1 + \|\psi_1\|^r_{L^{4r}} + \|\psi_2\|^r_{L^{4r}}) \\
&\le \quad C(1 + \|\nabla\psi_1\|^r_{L^2} + \|\nabla\psi_2\|^r_{L^2} + M^r),
\end{aligned}
$$

and due to (26) and (18), we obtain

$$
1 + \|\nabla\psi_2\|^2_{L^\infty} + \|f''(\xi)\|^{8/3}_{L^4} + \|\nabla\Delta\psi_1\|^2_{L^2} + \|\nabla v_2\|^{2p/(2p-d)}_{L^2} \in L^1(0,T),
$$

where we take into account that $2p/(2p-d) \le p$ is equivalent to the assumption $p \ge (d+2)/2$. Therefore, we can apply the Grönwall lemma and conclude $A(t) = 0$ for all $t > 0$. ∎

6 Regularity in two dimensions

In this section, we assume that

$$\alpha,\ m,\ \nu \text{ are positive constants, and } d = 2,$$

further, the viscous stress tensor is described by a differentiable functional, that is, there exists a strictly convex potential $U \in C^2(\mathbb{R}^4_{sym})$ of τ such that, for some $p \in (1, \infty)$ and positive constants γ_6 and γ_7,

$$\tau_{ij}(\zeta) = \frac{\partial U}{\partial \zeta_{ij}}(\zeta), \qquad U(0) = \frac{\partial U}{\partial \zeta_{ij}}(0) = 0 \tag{28}$$

$$\frac{\partial^2 U}{\partial \zeta_{ij} \partial \zeta_{kl}}(\zeta)\xi_{ij}\xi_{kl} \geq \gamma_6 (1 + |\zeta|)^{p-2}|\xi|^2 \tag{29}$$

$$|\frac{\partial^2 U}{\partial \zeta_{ij} \partial \zeta_{kl}}(\zeta)| \leq \gamma_7 (1 + |\zeta|)^{p-2}, \quad \forall \zeta, \xi \in \mathbb{R}^4_{sym}. \tag{30}$$

For $U(D(u)) = |D(u)|^2$, the system (1) reduces to the Navier-Stokes equations. When $f''(y) = O(|y|^r)$, $r < \infty$ as in Corollary 1, then $(u, \psi)(t) \in \mathbf{V}^p$ a.e. in time. We shall show that in fact, $(u, \psi)(t) \in \mathbf{V}^p$ for all $t > 0$ and the solution is then the unique strong solution.

Lemma 4 *Let (u, ψ) be the weak solution we have found for $p \geq 2$. If further $u_0 \in \mathbf{J}^{1,2}$ and (25) holds in the case $d = 2$, then $u \in L^\infty(0, T; \mathbf{J}^{1,2})$ and for any $0 < t < T$*

$$\begin{aligned} \|\nabla u\|^2_{L^2}(t) \quad + \quad \nu\gamma_6 \int_0^t \|\nabla^2 u\|^2_{L^2} &\leq \|\nabla u_0\|^2_{L^2} + \\ &+ C(1 + T + M^{4r} + Q(0)^{2r})Q(0)^2. \end{aligned} \tag{31}$$

Proof) Considering the Galerkin system (20)-(22) for the eigenvectors of the Stokes operator in $W^{2,2}(\Omega)^2$, we can multiply the i-th equation by $\lambda_i \xi_i(t)$, where λ_i are the corresponding eigenvalues, and sum over $i \in \mathbb{N}$. Hence, we can assume that u is smooth enough and do an apriori estimate. Multiplying the equation (1) by Δu and integrating it, we have

$$\frac{1}{2}\partial_t \int_\Omega |\nabla u|^2 + \nu\gamma_6 \int_\Omega (1+|\nabla u|)^{p-2}|\nabla^2 u|^2 \leq \alpha \int_\Omega |\Delta\psi||\nabla\psi||\Delta u|, \tag{32}$$

taking into account

$$-\int_\Omega (u \cdot \nabla) u \Delta u = \int_\Omega \partial_k u_j \partial_j u_i \partial_k u_i + \int_\Omega u_j \partial_{jk} u_i \partial_k u_i = 0.$$

Indeed, each term in the above equation vanishes for $d = 2$ using (2). For $p \geq 2$,

$$\frac{1}{2}\partial_t \int_\Omega |\nabla u|^2 + \nu\gamma_6 \int_\Omega |\nabla^2 u|^2 \quad \leq \quad C\|\Delta\psi\|_{L^2}^2 \|\nabla\psi\|_{L^\infty}^2 + \frac{\nu\gamma_6}{2}\|\nabla^2 u\|_{L^2}^2.$$

By (16), we have

$$\partial_t \int_\Omega |\nabla u|^2 + \nu\gamma_6 \int_\Omega |\nabla^2 u|^2 \leq C\|\nabla\psi\|_{L^2}^2 \|\nabla^3\psi\|_{L^2}^2.$$

Then (18), (26), and the above inequality imply (31), completing the proof. □

Lemma 5 *If $p > 1$, $u_0 \in \mathbf{J}^{1,2}$, $\psi_0 \in W^{2,2}$, $f \in C^3$, and f'', f''' both satisfy the growth condition (25) in case $d = 2$, then a solution for (1)-(3) satisfies*

$$\|\Delta\psi\|_{L^2}^2(t) + m\alpha \int_0^t \|\Delta^2\psi\|_{L^2}^2 \leq \|\Delta\psi_0\|_{L^2}^2 +$$
$$+ C[Q(0)^2 + (1 + T + M^{6r} + Q(0)^{3r})(Q(0) + Q(0)^2 + Q(0)^4)].$$

Moreover, $u \in L^\infty(0, T; \mathbf{J}^{1,2}) \cap L^2(0, T; \mathbf{J}^{2,p})$ for $p < 2$.

Proof) We apply Δ to (3) and multiply it by $\Delta\psi$. Then integrating it using divergence theorem, we have

$$\begin{aligned}
\partial_t \int_\Omega |\Delta\psi|^2 \quad &\leq \quad -2m\alpha \int_\Omega |\Delta^2\psi|^2 + \int_\Omega |u - M_u||\nabla\psi||\Delta^2\psi| \\
&\quad +2\int_\Omega |\Delta f'(\psi)||\Delta^2\psi| \\
&\leq \quad -m\alpha \int_\Omega |\Delta^2\psi|^2 + C\int_\Omega (|u - M_u|^2|\nabla\psi|^2 + |\Delta f'(\psi)|^2).
\end{aligned}$$

Here, $M_u = 1/|\Omega| \int_\Omega u$ is added since

$$\int_\Omega \Delta(M_u \cdot \nabla\psi)\Delta\psi = \frac{1}{2}\int_\Omega M_u \cdot \nabla|\Delta\psi|^2 = 0.$$

By (16) and the Poincaré inequality,

$$\begin{aligned}
\int_\Omega |u - M_u|^2|\nabla\psi|^2 \quad &\leq \quad \|u - M_u\|_{L^2}^2\|\nabla\psi\|_{L^\infty}^2 \leq C\|u\|_{L^2}^2\|\nabla\psi\|_{L^2}\|\nabla^3\psi\|_{L^2}, \\
\int_\Omega |\Delta f'(\psi)|^2 \quad &\leq \quad \int_\Omega (|f''(\psi)|^2|\Delta\psi|^2 + |f'''(\psi)|^2|\nabla\psi|^4) \\
&\leq \quad C(1 + \|\psi\|_{L^{2r}}^{2r})(\|\Delta\psi\|_{L^\infty}^2 + \|\nabla\psi\|_{L^\infty}^4) \\
&\leq \quad C(1 + M^{2r} + \|\nabla\psi\|_{L^2}^{2r})(\|\nabla\psi\|_{L^2}^{2/3}\|\nabla^4\psi\|_{L^2}^{4/3} \\
&\quad +\|\nabla\psi\|_{L^2}^{8/3}\|\nabla^4\psi\|_{L^2}^{4/3}).
\end{aligned}$$

Rearranging the terms and using (18), applying the Young inequality, and (26) we obtain

$$\partial_t \int_\Omega |\Delta\psi|^2 + m\alpha \int_\Omega |\Delta^2\psi|^2 \leq C(\|u\|_{L^2}^4 + \|\nabla\psi\|_{L^2}^2\|\nabla^3\psi\|_{L^2}^2)$$
$$+C(1 + M^{2r} + \|\nabla\psi\|_{L^2}^{2r})^3(\|\nabla\psi\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^8).$$

Integrating with respect to time, we recover the inequality. Then, since Δ commutes with the projection operators P_i^j, $i = 1, 2$, $j \in \mathbb{N}$, the solution obtained in Theorem 1 satisfies the required inequality.

To prove that $u \in L^\infty(0,T;\mathbf{J}^{1,2}) \cap L^2(0,T;\mathbf{J}^{2,p})$, we argue as in Lemma 4 to obtain (32). Applying the result given at [16, pp. 227] for $1 < p < 2$

$$\|\nabla^2 u\|_{L^p}^2 \leq C\mathcal{I}_p(u)(1 + \|\nabla u\|_p)^{2-p} \tag{33}$$

for some constant $C > 0$ and $\mathcal{I}_p(u) = \int_\Omega (1 + |\nabla u|)^{p-2}|\nabla^2 u|^2$. Hence it follows

$$\frac{1}{2}\partial_t \int_\Omega |\nabla u|^2 \quad + \quad \nu\gamma_6\mathcal{I}_p(u) \leq \alpha\|\Delta\psi\|_{L^{p'}}\|\nabla\psi\|_{L^\infty}\|\nabla^2 u\|_{L^p}$$
$$\leq \quad C\|\Delta\psi\|_{L^{p'}}\|\nabla\psi\|_{L^\infty}\mathcal{I}_p^{1/2}(u)(1 + \|\nabla u\|_p)^{(2-p)/2}.$$

By Hölder inequality, we have

$$\partial_t \int_\Omega |\nabla u|^2 + \nu\gamma_6\mathcal{I}_p(u) \quad \leq \quad C\|\Delta\psi\|_{L^{p'}}^2\|\nabla\psi\|_{L^\infty}^2(1 + \|\nabla u\|_p)^{2-p}.$$

Integrating in time this yields

$$\|\nabla u\|_{L^2}^2(t) + \nu\gamma_6 \int_0^t \mathcal{I}_p(u) \quad \leq \quad \|\nabla u_0\|_{L^2}^2 +$$
$$+ \quad C\int_0^t \|\Delta\psi\|_{L^{p'}}^{p'}\|\nabla\psi\|_{L^\infty}^{p'} + \int_0^t (1 + \|\nabla u\|_p)^p.$$

Then using the regularity of ψ, (19) and rewriting (33) as

$$\int_0^T \|\nabla^2 u\|_{L^p}^2 \leq (T + \sup_{t\in[0,T]} \|\nabla u\|_{L^2}^2(t)) \int_0^T \mathcal{I}_p(u),$$

the required result follows. ∎

As a direct consequence of the above two lemmas, the weak solution we have obtained satisfy almost everywhere the equations. From Lemmas 4 and 5 and Theorem 3, we can state the following regularity theorem.

Theorem 4 *For $p > 1$, $u_0 \in \mathbf{J}^{1,2}$, $\psi_0 \in W^{2,2}$, there exists a solution in*

$$\begin{array}{ll} L^\infty(0,T;\mathbf{J}^{1,2}\times W^{2,2})\cap L^2(0,T;\mathbf{J}^{2,2}\times W^{4,2}) & \text{if } p \geq 2 \\ L^\infty(0,T;\mathbf{J}^{1,2}\times W^{2,2})\cap L^2(0,T;\mathbf{J}^{2,p}\times W^{4,2}) & \text{if } p < 2 \end{array}$$

and the solution belongs to $\mathbf{V}^2$ as soon as $t > 0$ if $f \in C^3$ and f'', f''' both satisfy (25) in case $d = 2$. Moreover, this solution is unique if $p \geq 2$.

Proof) For $p \geq 2$, denoting by (u, ψ) the unique solution given by Theorem 3, Lemmas 4 and 5 guarantee that this solution belongs to $\mathbf{J}^{1,2} \times W^{2,2}$ for any $t > 0$. For any $0 < t_1 < t_2$, $u(t)$, $\nabla\psi(t) \in (L^q)^2$, $\forall q > 1$ and $\psi(t) \in L^\infty$ uniformly with respect to $t \in [t_1, t_2]$. Thus

$$\partial_t\psi + \Delta^2\psi = u\cdot\nabla\psi + \Delta f'(\psi) \equiv h \in L^2(t_1,t_2;L^2).$$

Therefore, $\psi \in W^{3,2}$ for $t_1 < t < t_2$, which finishes the proof for $p \geq 2$.

For $1 < p < 2$, arguing as in the proof of Theorem 2 with $k = 2$, we obtain an approximate solution (u^i, ψ^i, μ^i) $i \in \mathbb{N}$ such that, for any $T > 0$, $(u^i, \psi^i) \in L^\infty(0,T;\mathbf{H})$, $\nabla u^i \in L^p(0,T;L^p)$, $\partial_t\psi^i \in L^2(0,T;H^{-1})$, $\nabla\mu^i \in L^2(0,T;L^2)$, and $\mu^i \in L^2(0,T;W^{1,2})$ uniformly with respect to $i \in \mathbb{N}$; and $\partial_t u^i$ belongs to a bounded set of

$$L^\gamma(0,T;(W^{2,2}\cap\mathbf{J}^{1,p})')$$

with $\gamma = \min(p, 2(p-1))$. Indeed,

$$\begin{aligned} |\langle\partial_t u^i, v\rangle| &\leq \|u^i\|_{L^{p'}}\|\nabla u^i\|_{L^p}\|P_1^i v\|_{L^\infty} \\ &\quad +\gamma_2\gamma_7(1+\|\nabla u^i\|_{L^p})^{p-1}\|\nabla P_1^i v\|_{L^p} \\ &\quad +\alpha\|\Delta\psi\|_{L^2}\|\nabla\psi\|_{L^2}\|P_1^i v\|_{L^\infty} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Let us estimate I_1, I_2 and I_3 separately. Due to the interpolation inequality(see [16, pp. 232], for instance),

$$\|u\|_{L^{p'}} \leq \|u\|_{L^2}^{\frac{3p-4}{2(p-1)}}\|u\|_{L^{\frac{2p}{2-p}}}^{\frac{2-p}{2(p-1)}},$$

we have

$$\begin{aligned} \int_0^T I_1 dt &\leq \|u^i\|_{L^\infty(0,T;L^2)}\int_0^T \|\nabla u^i\|_{L^p}^{1+\frac{2-p}{2(p-1)}}\|P_1^i v\|_{W^{2,2}}dt \\ &\leq C\|u^i\|_{L^p(0,T;\mathbf{J}^{1,p})}^{\frac{1}{2(p-1)}}\|v\|_{L^{\frac{2(p-1)}{2p-3}}(0,T;W^{2,2})}. \end{aligned}$$

The second one yields

$$\begin{aligned}\int_0^T I_2 dt &\leq C\int_0^T (1+\|\nabla u^i\|_{L^p})^{p-1}\|v\|_{W^{2,2}}dt \\ &\leq C\left(1+\|u^i\|^{p-1}_{L^p(0,T;\mathbf{J}^{1,p})}\right)\|v\|_{L^p(0,T;W^{2,2})}\end{aligned}$$

taking into account that $(W^{2,2})^2 \hookrightarrow (W^{1,p})^2$ if $p > 1$.

From Lemma 5, we obtain

$$\int_0^T I_3 dt \leq C\|\Delta\psi\|_{L^\infty(0,T;L^2)}\|\nabla\psi\|_{L^\infty(0,T,L^2)}\int_0^T \|v\|_{W^{2,2}}dt.$$

Hence, $\partial_t u^i \in L^\gamma(0,T;((W^{2,2})^2 \cap \mathbf{J}^{1,p})')$, $u^i \in L^2(0,T;\mathbf{J}^{2,p})$ $p < 2$ and $\mathbf{J}^{2,p} \hookrightarrow\hookrightarrow (W^{1,p})^2$, $p \geq 1$, then

$$\nabla u^i \to \nabla u \text{ a.e. in } \Omega \times (0,T)$$

and also (since $\tau_{ij} \in C^1(\mathbb{R}^4_{sym})$)

$$\tau(D(u^i)) \to \tau(D(u)) \text{ a.e. in } \Omega \times (0,T).$$

By standard arguments [16, pp. 224], this implies

$$\int_0^T\int_\Omega \nu^i\tau^i : D(v) \to \int_0^T\int_\Omega \nu\tau : D(v)dxdt.$$

$\square$

7 Non-differentiable case

In order (1)-(3) to model a phase transition phenomena, since $\psi(x,t)$ indicates the phase of the system at (x,t), it is required that $|\psi| \leq 1$. In this section, we show that $|\psi| \leq 1$ may be obtained by using a standard penalization scheme for a non-differentiable free energy [2, 11]. Let $f = f_1 + f_2$; f_1 satisfies (5) and

$$f_2(y) = \begin{cases} 0 & \text{for } |y| \leq 1 \\ +\infty & \text{for } |y| > 1. \end{cases}$$

The subdifferential of f_2 is denoted by ∂f_2, and we set

$$f'(y) = \{f_1'(y) + \chi | \chi \in \partial f_2(y)\}.$$

Definition 3 *We say that* (u, ψ, μ) *is a generalized solution for (1)-(3) if for any* $(v, \phi) \in \mathbf{V}^p$ *with* $f_2(\phi) \in L^1$, (u, ψ, μ) *satisfy the definition of a weak solution with (13) replaced with*

$$\int_\Omega \big((\mu - f_1'(\psi))(\psi - \phi) - \sqrt{\alpha}\nabla\psi \cdot \nabla(\sqrt{\alpha}(\psi - \phi))\big) \\ \geq \int_\Omega (f_2(\psi) - f_2(\phi)) \quad a.e.\ t. \qquad (34)$$

Theorem 5 *Given* $(u_0, \psi_0) \in \mathbf{H}$ *with* $|\psi_0| \leq 1$ *and* f *as above, for* $p \geq (3d+2)/(d+2)$ *if* $d = 2, 3$ *or in the Newtonian case* $\tau = D(u)$ *with* $p = 2$ *and* $d = 3$*, there exists a generalized solution of (1)-(3) provided*

$$(1 + |y|)|\alpha'(y)| \leq 2\alpha(y).$$

Proof) We first introduce the following approximating sequence of f_2:

$$\begin{array}{ll} f_2^j(y) = 0 & \text{for } |y| < 1 \\ f_2^j(y) = j(|y|^2 - 1)^2 & \text{for } |y| \geq 1. \end{array}$$

Now, we consider the initial boundary problem (1)-(3) with $f^j \equiv f_1 + f_2^j$. There exists a weak solution corresponding to f^j by Theorem 1. We denote by (u^j, ψ^j, μ^j) the corresponding solutions. Since $Q(u_0, \psi_0)(f^j) \leq Q(u_0, \psi_0)(f) < \infty$, repeating the argument of Theorem 1, we deduce that (u^j, ψ^j), $j \in \mathbb{N}$ is a bounded sequence in $L^\infty(0, T; \mathbf{H})$, u^j in $L^p(0, T; \mathbf{J}^{1,p})$, and μ^j in $L^2(0, T; W^{1,2})$ for any $T > 0$. Also, $\partial_t u^j \in L^1(0, T; (\mathbf{J}^{1,p})')$ and $\partial_t \psi^j \in L^2(0, T; H^{-1})$ uniformly with respect to j by (23) and (24). Thus, as $j \to \infty$ a subsequence of (u^j, ψ^j, μ^j) converges to (u, ψ, μ) strongly in $L^2(0, T; \mathbf{J}^{0,2} \times L^2 \times L^2)$ and weakly in $L^p(0, T; \mathbf{J}^{1,p}) \times L^2(0, T; W^{1,2} \times W^{1,2})$.

It is easy to show that (u, ψ, μ) satisfies (11) and (12) as in Theorem 1. Due to the lower semi-continuity of norms and the Fatou's lemma, (u, ψ, μ) satisfies (18) with f. As a consequence, $|\psi| \leq 1$ for all $t \in [0, T]$. Finally, we show (34). Without loss of generality, we can assume $f_1'(y) = O(|y|^2)$ as in the proof of Theorem 1. Then, for $\phi \in W^{3,2}$ with $f(\phi) \in L^1$,

$$\begin{aligned} \int_\Omega (\mu - f_1'(\psi))(\psi - \phi) &= \lim_j \int_\Omega (\mu^j - f_1'(\psi^j))(\psi^j - \phi) \ \ a.e.\ t \\ \int_\Omega (f_2(\psi) - f_2(\phi)) &\leq \liminf_j \int_\Omega (f_2^j(\psi^j) - f_2^j(\phi)) \ \ a.e.\ t. \end{aligned}$$

We notice $(\alpha(\psi^j) + (\psi^j - \phi)\alpha'(\psi^j))|\nabla\psi^j|^2$ is weakly lower semi-continuous since ψ^j converges strongly in L^2 a.e. t and $\alpha(\psi^j) + (\psi^j - \phi)\alpha'(\psi^j) \geq 0$

by the assumption. Therefore,

$$\begin{aligned}
&\liminf_j \int_\Omega \sqrt{a(\psi^j)}\nabla\psi^j \cdot \nabla(\sqrt{a(\psi^j)}(\psi^j - \phi)) \\
&= \liminf_j \int_\Omega (\alpha(\psi^j) + \alpha'(\psi^j)(\psi^j - \phi))|\nabla\psi^j|^2 - \int_\Omega \alpha\nabla\psi\nabla\phi \\
&\geq \int_\Omega (\alpha + \alpha'(\psi - \phi))|\nabla\psi|^2 - \int_\Omega \alpha\nabla\psi\nabla\phi \quad a.e.\ t.
\end{aligned}$$

This proves (34) and completes the proof. ∎

Finally, we also obtain the existence of a measure-valued solution in the following theorem, for $p \geq 2(d = 2, 3)$.

Theorem 6 *Under the assumptions of Theorem 5, there exists a measure-valued solution* (u, λ, ψ, μ) *satisfying Definition 2 with (13) replaced with (34).*

Proof) The proof is similar to the one of Theorem 5, replacing $\partial_t u^j \in L^1(0, T; (\mathbf{J}^{1,p})')$ by $\partial_t u^j \in L^{p'}(0, T; (\mathbf{J}^{k,2})')$, $k > 1 + d/2$, and arguing as in Theorem 2. ∎

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