Huygens' Principle for Hyperbolic Operators and Integrable Hierarchies

Fabio A. C. C. Chalub[1] and Jorge P. Zubelli[2]

November 3, 2004

[1]Centro de Matemática e Aplicações Fundamentais, Av. Prof. Gama Pinto, 2, P-1649-003, Lisboa, Portugal. chalub@cii.fc.ul.pt. Supported by FCT/Portugal through the Project FCT-POCTI/34471/MAT/2000.

[2]IMPA, Est. D. Castorina 110, RJ 22460-320, Rio de Janeiro, Brazil. zubelli@impa.br.
Supported by CNPq, Brazil, through grants number 30.1003 and 200258.

Abstract

We show that the stationary solutions of the canonical AKNS hierarchy of non-linear evolution equations yield perturbations of Dirac operators that satisfy a strict form of Huygens' principle. Namely, the domain of dependence of such Dirac operators at a given point y is contained in the light-cone's hypersurface issued from y. By canonical AKNS hierarchy we mean that the differential polynomials defining the flows are isobaric with respect to certain weights. The method we employ is of interest by itself. Indeed, we consider the Riesz kernels associated to a given hyperbolic differential operator and expand the fundamental solution of perturbations of this operator in a series in such Riesz kernels. Using the coefficients of this Hadamard type expansion we introduce a family of vector fields. For the D'Alembertian such vector field family corresponds to the KdV hierarchy and for the Dirac operators they include the AKNS one.

Keywords: Huygens' principle, Dirac operators, Rational Solutions of Integrable Equations.

1 Introduction

The Korteweg-de Vries (KdV) equation

$$u_t = 6uu_x - u_{xxx} \tag{1.1}$$

and the cubic nonlinear Schrödinger equation (NLS) are prototypes of completely integrable systems in infinite dimensions. Such complete integrability comes as part of a full package of surprising connections among seemingly unrelated areas ranging from spectral theory of differential operators to Liouville integrable particle systems [10, 28, 29].

The complete integrability of (1.1) is intrinsically connected to the fact that it is one of the flows of an infinite hierarchy of commuting Hamiltonian flows canonically associated to the resolvent expansion of a Schrödinger operator of the form $-\partial_x^2 + u$. In a similar fashion, the NLS is connected with a more general hierarchy which is known as the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy [1]. Here again, we have the spectral theory of a certain operator playing a crucial role. See [14, 15].

The present work is concerned with yet another surprising connection between completely integrable systems and the strict form of Huygens' principle. The latter consists of an interesting situation where the support of the fundamental solution $\Phi(x; y)$ to a hyperbolic differential operator $\mathcal{L} = \mathcal{L}(x, \partial_x)$ is contained in the surface of the light-conoid associated to the principal part of $\mathcal{L}$. See [19, 13]

We shall refer, henceforth, to operators satisfying Huygens' principle in its strict form as operators of *Huygens' type* or say they satisfy *Huygens' principle*. An example of operator where Huygens' principle holds is the wave-operator $\Box \stackrel{\text{def}}{=} \partial_0^2 - \sum_{i=1}^n \partial_i^2$, whenever $n > 1$ is an odd integer. It does not hold, however, if n is even or $n = 1$. Another important example, for odd n, is the free (massless) Dirac operator

$$\partial\!\!\!/ \stackrel{\text{def}}{=} \sum_{\mu=0}^{n} \gamma^\mu \partial_\mu \ , \tag{1.2}$$

where $\{\gamma^\mu\}_{\mu=0}^n$ is the set of Dirac matrices associated to the Minkowski pseudo-metric $((g^{\mu\nu}))_{\mu,\nu=0}^n = \mathrm{diag}[1, -1, \cdots, -1]$ and satisfying the Clifford relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I \qquad \mu, \nu = 0, \cdots, n \ . \tag{1.3}$$

The surprising connection alluded above is the fact that large classes of Huygens' type operators can be constructed perturning $\Box$ by special solutions of completely integrable hierarchies of nonlinear evolution equations. One such example is provided by the rational solutions of the Korteweg-de Vries (KdV) hierarchy decaying at infinity. By that we mean, rational functions that remain rational when evolved by the equations of the KdV hierarchy, and that decay at infinity. They were subject of extensive investigation in now classical papers [2, 3, 9]. Such family of Huygens' type operators were found by Lagnese and Stellmacher as counter-examples to a conjecture attributed to Hadamard. Lagnese and Stellmacher, in fact, were not aware of the connection with soliton theory, which at the time of [21] was at its infancy. This connection was made clear only in the reference [25]. In soliton language, they showed that if $u(\cdot)$ is a stationary rational solution of the KdV hierarchy decaying at infinity, then for sufficiently large odd n the operator

$$\mathcal{L} = \Box - u(x^0) \tag{1.4}$$

is of Huygens' type. Furthermore, under an analyticity requirement on $u(\cdot)$, they showed that if $\mathcal{L}$ in (1.4) is Huygens, then it is necessarily in the class of rational solutions of the KdV hierarchy decaying at infinity.

Another surprising connection was found by Y. Berest in the context of soliton solutions of the KdV hierarchy and two-variable perturbed $\Box$. See [7].

In a previous work [8], we generalized the result of Lagnese-Stellmacher by showing that any pair $(q, r) = (q(x^0), r(x^0))$ in a certain family of rational solutions of the AKNS hierarchy yields perturbations of Dirac operators of Huygens' type. More precisely, the operator

$$\mathcal{D} \stackrel{\text{def}}{=} \partial\!\!\!/ - \frac{q+r}{2} I + \frac{q-r}{2} \bar{\gamma} \quad , \tag{1.5}$$

where $\bar{\gamma} \stackrel{\text{def}}{=} (-1)^{(n-1)/4} \gamma^0 \gamma^1 \cdots \gamma^n$, turns out to be of Huygens' type, for sufficiently large odd n.

Our main result, in the present article, deepens such connection by showing that if $(q(x^0), r(x^0))$ is a stationary solution of the (canonical) AKNS hierarchy, then $\mathcal{D}$ as in (1.5) is also of Huygens' type. To prove this result, we rely on the Hadamard's ansatz for the fundamental solution $\Psi(x; y)$ of the Dirac operator $\mathcal{D}$ of the form

$$\Psi(x; y) = \sum_{k=0}^{\infty} \left\{ \Theta^{\alpha_0 + 2k}(x - y) s_{2k}(x, y) + \Lambda^{\alpha_0 + 2k}(x - y) s_{2k+1}(x, y) \right\} ,$$

where as defined below Λ^α and $\Theta^\alpha = \partial\!\!\!/ \Lambda^\alpha$ are Riesz kernels of the operator $\partial\!\!\!/$ and $\alpha_0 = -n + 1$. The approach to prove our main result is of interest by itself since it relies on a connection between the coefficients $(s_k)_{k=0}^{\infty}$ and the AKNS hierarchy. Furthermore, we show that the series for Ψ terminates iff (q, r) are stationary solution of the AKNS hierarchy.

In the final section, we introduce the concept of a *Hadamard's hierarchy*. We show that for operators of the form (1.5) the Hadamard's hierarchy coincides with the canonical AKNS hierarchy for (q, r). This conjoined with the fact that for wave type operators as in (1.4) the Hadamard's hierarchy coincides with the KdV hierarchy leads to the problem of classifying Hadamard's hierarchies according to their integrability.

The structure of this article goes as follows:

In the remaining part of this section, we recall the construction of the Hadamard's series. In Section 2, we recall the definition of Huygens's principle, and the construction of the AKNS and KdV hierarchies. Then, we discuss what we call the *canonical* AKNS hierarchy.

In Sections 3 and 4 we prove, for wave operators and for Dirac operators, respectively, our main results and show how to apply them to construct examples of operators of Huygens type.

Finally, in Section 5 we introduce the concept of Hadamard's hierarchy and show how it reduces to the KdV and AKNS hierarchies in the wave and Dirac operator case. We also show examples of matrix-wave operators of Huygens type which seem to appear for the first time in the literature. This also allows us to suggest a matrix generalization of the Miura transformation.

1.1 The Hadamard's Series

Let $\mathcal{L} = \mathcal{L}(x, \partial_x)$ be a hyperbolic differential operator of order κ acting on vector valued functions of the variable $x = (x^0, x^1, \cdots, x^n) \in \mathbb{R}^{n+1}$. We set the pseudo-norm $|| \cdot ||_{\mathcal{L}}$ to be the one induced by the principal part of $\mathcal{L}$. See [19]. For a nonempty open set $\mathcal{O} \subset \mathbb{R}^{n+1}$ we assume that there exists

$$\mathbb{C} \ni \alpha \longmapsto \mathcal{R}^{\alpha}(\cdot, \cdot\cdot) \in \mathcal{D}'(\mathcal{O} \times \mathcal{O}) \ ,$$

a holomorphic family of distributions such that for some $\alpha_0 \in \mathbb{Z}$ we have

- $\mathcal{R}^{\alpha_0 - \kappa}(x, y) = \delta_y(x)$,
- $\mathcal{L}\mathcal{R}^{\alpha} = \mathcal{R}^{\alpha - \kappa}$.

We call $\mathcal{R}^{\alpha}$ a *Riesz kernel* associated to $\mathcal{L}$.

It is immediate that $\mathcal{R}^{\alpha_0}$ is a fundamental solution of $\mathcal{L}$. See [4, 5] for motivation and examples. We have that $\mathcal{L}$ possesses Huygens' property in $\mathcal{O}$ iff

$$\mathrm{supp}\mathcal{R}^{\alpha_0}(\cdot, y) \subset \{x \in \mathbb{R}^{n+1} \big| ||x - y||_{\mathcal{L}} = 0\} \ , \forall y \in \mathcal{O} \ .$$

Now, consider perturbations of $\mathcal{L}$ of the form $\mathcal{L} + u(x)$ and its fundamental solution $\Psi(x; y)$, i.e.,

$$(\mathcal{L} + u(x))\Psi(x; y) = \delta_y(x) \ . \tag{1.6}$$

with $\Psi(x; y)$ given by the formal series expansion

$$\Psi(x; y) = \sum_{k=0}^{\infty} \mathcal{R}^{\alpha_0 + \kappa k}(x, y) s_k(x, y) \ .$$

The above equation holds if u is analytic or if the formal series expansion is a finite sum [6]. In most of what follows, we shall consider analytic perturbations and therefore fall within such framework. The results that will follow are certainly extensible to more general instances but we will not dwell on this matter here. The coefficients s_k are called *Hadamard's coefficients* and $(s_k)_{k=0}^{\infty}$ is called *Hadamard's sequence.*

The requirement (1.6) imposes that the coefficients s_k obey a certain recursion that could be explicitly obtained in some particular cases. Such recursion will be referred to as *Hadamard's recursion.* In general the Hadamard's coefficients are uniquely determined imposing their regularity in the vicinity of the vertex of the light-cone, i.e., when $y \to x$.

2 Preliminary Results and Background

In this section we collect a few preliminary results that will be used in the sequel. Subsection 2.1 concerns transformations that trivially preserve Huygens' property for Dirac type operators. Subsection 2.2 deals with a construction of the canonical flows of the AKNS hierarchy that, albeit now standard, is crucial for some of the results of the present article. We omit the motivation for the AKNS construction and refer the interested reader to the literature [1, 23].

2.1 Huygens Preserving Transformations

It is natural to seek transformations that trivially preserve Huygens' property. We focus on the Dirac operators. One can show that the following transformations preserve Huygens property [8]:

1. Change of independent variables by a smooth diffeomorphism:

   $\tilde{x}^\mu = f^\mu(x^0, \cdots, x^n),\ \mu = 0, \cdots, n$, with $\det\big(\partial_\mu f^\nu\big)_{\mu,\nu=0,\cdots,n} \neq 0$.

2. Left multiplication:

   Take $\mathcal{D} \mapsto \bar{\mathcal{D}} = \Xi(x)\mathcal{D}$ and $\Psi \mapsto \bar{\Psi} = \Psi\Xi(y)^{-1}$, where $\Xi \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{N\times N})$ and $\Xi(x)$ is a non-singular matrix for all x.

3. Factor transformations:

   Let ρ be a non-singular smooth matrix-valued function, i.e.,

   $$\rho = \rho_\phi I + \rho_\mu \gamma^\mu + \bar{\rho}_\mu \gamma^\mu \bar{\gamma} + \bar{\rho}\bar{\gamma} \ ,$$

   where ρ_ϕ, ρ_μ, $\bar{\rho}_\mu$ and $\bar{\rho}$ are smooth functions with $\det(\rho(x)) \neq 0$, for all x in the domain under consideration. The factor transformation consists in the mapping $\mathcal{D} \mapsto \bar{\mathcal{D}} = \rho(x)\mathcal{D}\rho(x)^{-1}$ and $\Psi \mapsto \bar{\Psi} = \rho(x)\Psi\rho(y)^{-1}$.

Two Dirac operators that are related by means of compositions of the above transformations will be called *trivially equivalent.*

2.2 The AKNS and the KdV Hierarchies

Let us recursively define three sequences of differential polynomials $\{e_l\}_{l=0}^{\infty}$, $\{f_l\}_{l=0}^{\infty}$, and $\{h_l\}_{l=0}^{\infty}$ on the functions $q = q(x)$ and $r = r(x)$ as follows:
For $l = 0, 1, \cdots$ we set

$$e_{l+1} = qh_l + \frac{1}{2}\partial_x e_l \ , \tag{2.1}$$

$$f_{l+1} = rh_l - \frac{1}{2}\partial_x f_l \ , \tag{2.2}$$

$$h_{l+1} = -\frac{1}{2} \sum_{\substack{m+n=l+1 \\ m,n\geq 1}} (e_n f_m + h_n h_m) \tag{2.3}$$

with

$$e_0 = f_0 = 0 \quad h_0 = 1 \ . \tag{2.4}$$

Definition 2.1. *The system of (nonlinear partial differential) equations*

$$\left\{ \begin{array}{rcl} q_{t_\ell} & = & 2e_{\ell+1}(q,\ldots,\partial_x^\ell q; r,\ldots,\partial_x^{\ell-1} r) \\ r_{t_\ell} & = & -2f_{\ell+1}(q,\ldots,\partial_x^{\ell-1} q; r,\ldots,\partial_x^{\ell} r) \ . \end{array} \right. \tag{2.5}$$

will be called the ℓ-th equation (or the ℓ-th flow) of the AKNS hierarchy.

One motivation of the above construction, which will be relevant to the sequel, is the following: Let

$$R^{(\ell)} \stackrel{\text{def}}{=} (R_0 k^\ell + R_1 k^{\ell-1} + \cdots + R_\ell) \ , \tag{2.6}$$

where

$$R_l \stackrel{\text{def}}{=} e_l E + f_l F + h_l H \tag{2.7}$$

with

$$H = \mathrm{diag}[1,-1] \quad \text{and} \quad E = F^\top = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ .$$

One can show [23, 30] that equation (2.5) implies the compatibility condition of

$$\partial_x \Psi = (kH + Q)\Psi \stackrel{\text{def}}{=} (kH + qE + rF)\Psi \ , \tag{2.8}$$

and

$$\partial_{t_\ell}\Psi = R^{(\ell)}\Psi \ . \tag{2.9}$$

It is well-known that the flows generated by (2.5) commute with one another and thus it makes sense to consider q and r as functions of arbitrarily many time variables $t_1, t_2, \cdots$. In fact, they are Hamiltonian vector fields whose corresponding Hamiltonian are in involution with one another [1, 23]. Furthermore, the AKNS hierarchy comes as the compatibility condition of equations (2.8) and (2.9) for arbitrary ℓ. Such compatibility condition can be expressed for $l = 0, 1, \cdots$

$$
\begin{aligned}
e_{l+1} &= qh_l + \frac{1}{2}\partial_x e_l \ , && (2.10)\\
f_{l+1} &= rh_l - \frac{1}{2}\partial_x f_l \ , && (2.11)\\
\partial_x h_{l+1} &= f_{l+1}q - e_{l+1}r \ . && (2.12)
\end{aligned}
$$

Remark 2.1. *The recursion (2.1–2.3) is one possible solution of such recursive system of equations. In fact, one can characterize the recursion (2.1–2.3) as the only such solution whose term h_k is an isobaric differential polynomial on q and r with weight k if we give $\partial_x^l q$ and $\partial_x^l r$ weight $l+1$. Also e_k and f_k are isobaric of weight k. In this sense, we shall refer to Equation (2.5) with (e_ℓ, f_ℓ) as in (2.1–2.3) as the* canonical *AKNS hierarchy. Other choices of the integration constant in Equation (2.12) lead to linear combinations of the k-th canonical flow with lower orderflows [23, 11].*

Lemma 2.1. *For $k \geq 1$, the canonical h_k is a differential polynomial with none of its monomials depending only on q or r, i.e.,*

$$
h_k = \sum_{\substack{\sum_i (m_i+1)n_i + (m_i'+1)n_i' = k \\ \sum_i n_i > 0, \sum_i n_i' > 0}} \alpha^k_{m_i,n_i,m_i',n_i'} \prod_{i\geq 0} (q^{(m_i)})^{n_i} (r^{(m_i')})^{n_i'} \ ,
$$

with $\alpha^k_{m_i,n_i,m_i',n_i'} \in \mathbb{R}$.

Proof. Suppose $q = 0$ and r arbitrary. Then, from Equation (2.1) we find $e_{l+1} = \frac{1}{2}\partial_x e_l$, $e_0 = 0$ and then $e_k = 0$, $\forall k$. From Equation (2.3), we write

$$
h_{l+1} = -\frac{1}{2} \sum_{\substack{m+n=l+1 \\ m,n\geq 1}} h_n h_m
$$

and we prove that $h_k = 0$, $k \geq 1$, by induction in k, starting from $k = 1$. So, $h_{k+1} = h_{k+1}[q, r, q^{(1)}, r^{(1)}, \cdots, q^{(k-1)}, r^{(k-1)}]\big|_{q\equiv 0}$ cannot have any monomial depending on $r, r^{(1)}, r^{(2)}, \cdots, r^{(k-1)}$. The same holds true if q is arbitrary and $r = 0$. □

Lemma 2.2. *Given two different solutions of recursion (2.10–2.12), (e_k^1, f_k^1, h_k^1) and (e_k^2, f_k^2, h_k^2), there exists constants d_k, $k \geq 0$, with $d_0 = 1$, such that*

$$
(e_k^2, f_k^2, h_k^2) = \sum_{i=0}^{k} d_{k-i}(e_i^1, f_i^1, h_i^1) \ .
$$

Proof. Let (e_k^0, f_k^0, h_k^0) be the canonical AKNS. Then both (e_k^j, f_k^j, h_k^j), $j = 1, 2$ can be written as (see [16])

$$
(e_k^j, f_k^j, h_k^j) = \sum_{i=0}^{k} d_{k-i}^j (e_i^0, f_i^0, h_i^0) \ , \quad d_0^j = 1 \ .
$$

□

Remark 2.2. *If, in the above construction, we impose $r = 1$, and restrict ourselves to the odd flows, we have the KdV hierarchy. If we start from the canonical AKNS, then we shall call such particularization* the canonical KdV hierarchy.

An important class of solutions of the AKNS hierarchy is characterized by the fact that all the sufficiently high flows of such hierarchy vanish. This is the case, for instance, of the rational solutions of the AKNS studied in [31, 24]. This leads naturally to the following:

Definition 2.2. *We say that (q, r) is a stationary solution of the AKNS hierarchy if there exists $k_0 \geq 1$ such that $e_k[q, r] = f_k[q, r] = 0$ for all $k \geq k_0$.*

3 Huygens' Principle for Matrix Wave Operators

We start by defining a family of Riesz kernels for $\Box$. See [6, 12] for details on such construction. For $\Re\alpha > 0$ we define $\Lambda^\alpha = N(\alpha)\lambda_+^\alpha$ with

$$\lambda_+^\alpha = \begin{cases} [(x^0)^2 - \sum_{i=0}^n (x^i)^2]^{\alpha/2} & \text{for } (x^0)^2 \geq \sum_{i=0}^n (x^i)^2 \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

$$N(\alpha) = \frac{1}{2}\left[2^{\alpha+n}\pi^{(n-1)/2}\Gamma\left(\frac{\alpha+n+1}{2}\right)\Gamma\left(\frac{\alpha}{2}+1\right)\right]^{-1} , \tag{3.2}$$

and extend such family as a holomorphic family of distributions in $\mathcal{D}'(\mathbb{R}^{n+1})$ for all $\alpha \in \mathbb{C}$. $N(\alpha)$ obeys the recursion

$$(\alpha + 2)(\alpha + n + 1)N(\alpha + 2) = N(\alpha) . \tag{3.3}$$

For $\alpha = -2, -4, -6, \cdots$ or $\alpha = -n-1, -n-3, \cdots$, $\Lambda^\alpha(\cdot; y) \subset \mathcal{C}(y)$.

Let us consider an operator of the form

$$\Box \otimes I + u ,$$

where I is the identity matrix and u is a given matrix potential. Then, a formal fundamental solution is given by

$$\Psi(x; y) = \sum_{k=0}^{\infty} \Lambda^{-n+1+2k}(x - y)w_k(x, y) ,$$

where the Hadamard's coefficients w_k obey the recursion

$$\begin{aligned} w_0 &= I , \\ w_k + \frac{1}{k}(x^\mu - y^\mu)\partial_\mu w_k &= -(\Box \otimes I + u)w_{k-1} . \end{aligned} \tag{3.4}$$

In the notation of Subsection 1.1, $\alpha_0 = -n + 1$, $\kappa = 2$, and $\mathcal{R}^\alpha(x; y) = \Lambda^\alpha(x - y)$.

The properties of the Riesz kernels Λ^α give us, immediately, the following

Theorem 3.1. *Let $n > 1$ be an odd integer. If the Hadamard's coefficients in the expansion (3.4) of the operator $\Box \otimes I + u$ vanish $\forall k \geq k_0 \stackrel{def}{=} (n-1)/2$ then the operator is of Huygens' type.*

The condition that $w_k = 0$ for every $k \geq k_0$ is crucial for all that follows and it motivates us to say that the Hadamard's expansion is *terminating* or *truncating* at $k_0 - 1$.

The relations between Huygens' principle for scalar wave operator and the KdV Hierarchy can be established by means of the following result: [1]

Theorem 3.2. *Assume that u is a scalar function depending only on the time variable x^0 and let $W_k(x^0)$ denote the Hadamard's coefficient $w_k(x,y)$ of Equation (3.4) along the diagonal $y = x$. Then, for $k = 1, 2, \cdots$, $W_k(x^0) = W_k[u](x^0)$ is a differential polynomial in u such that*

$$\mathbf{X}_k[u](x^0) \stackrel{def}{=} \partial_{x^0} W_k[u](x^0) \ , \tag{3.5}$$

is the k-th vector field of the canonical KdV hierarchy.

Proof. First notice that because of Equation (3.4), the assumption that u depends only on x^0, and the fact that the Hadamard's coefficients must be bounded when $x \to y$ one can show by induction that the quantity $w_k(x,y)$ only depends on (x^0, y^0). Now we set

$$W_k^p \stackrel{\text{def}}{=} \lim_{y^0 \to x^0} \partial_0^p w_k(x^0, y^0) \ .$$

and differentiate p times the second equation in (3.4), in the scalar case, with respect to x^0. Upon taking the limit $y^0 \to x^0$ we get

$$\frac{k+p}{k} W_k^p = -W_{k-1}^{p+2} - \sum_{i=0}^{p} \binom{p}{i} u^{(p-i)} W_{k-1}^i \ . \tag{3.6}$$

Equation (3.6) defines a recursion with initial condition given by $W_0^p = \delta_0^p$, where $\delta_0^0 = 1$ and $\delta_0^p = 0$ for $p \neq 0$ is the Kronecker delta. Modulo numerical constants, this is equivalent to the linear recursion relation satisfied the KdV-hierarchy differential polynomials $H_k^p(x^0) = (-1)^k W_k^p(x^0)/k!$ introduced in the proof of Theorem 5.3 in [26]. $\Box$

4 Huygens' Principle for Dirac Operators

Let $((g^{\mu\nu})) = \text{diag}[1, -1, \cdots, -1]$ denote the Minkowski tensor. Associated to $g^{\mu\nu}$ we construct a Clifford Algebra. It is an associative algebra (with identity I) over the reals generated by all linear combinations of the form

$$(\gamma^0)^{m_0} (\gamma^1)^{m_1} \cdots (\gamma^n)^{m_n}, \quad m_\mu \in \{0, 1\} \ ,$$

[1]F.A.C.C.C. thanks Prof. Schimming for mentioning such result.

where the matrices $\{\gamma^\mu\}$, obey the relation $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I$. The Dirac matrices $\{\gamma^\mu\}$, are linearly independent [18, 22]. They satisfy

$$(\gamma^0)^\dagger = \gamma^0 \ , \quad (\gamma^i)^\dagger = -\gamma^i \ , \quad (\bar\gamma)^\dagger = \bar\gamma \stackrel{\text{def}}{=} (-1)^{(n-1)/4}\gamma^0\gamma^1\cdots\gamma^n \ . \tag{4.1}$$

Dirac operators are defined by [27]:

$$\mathcal{D} = \gamma^\mu\partial_\mu + v \ ,$$

where the summation for repeated indexes is implied. We shall adopt the notation $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ and restrict ourselves to the case where v is a linear combination of I and $\bar\gamma$. It is easy to see that

$$\partial\!\!\!/^2 = \Box \otimes I \ . \tag{4.2}$$

For odd n (and only in this case), $\bar\gamma$ is uniquely defined, modulo a sign, from the relations $\bar\gamma\gamma^\mu + \gamma^\mu\bar\gamma = 0$, $\mu = 0, 1, \cdots, n$ and $\bar\gamma^2 = I$.

We recall a fundamental solution of the Dirac operator is $\Psi(\cdot;\cdot) \in \mathcal{D}'(\mathcal{O}\times\mathcal{O})$ such that

$$(\partial\!\!\!/ + v)\Psi(\cdot; y) = \delta_y \ ,$$

where δ_y denotes the Dirac-delta distribution supported in an arbitrary point y in space-time.

From Equation (4.2) we can see that $\partial\!\!\!/\Lambda^\alpha$ is a fundamental solution of the free Dirac operator $\partial\!\!\!/$ whenever Λ^α is a fundamental solution of wave operator.

This motivates the definition of a family of distributions

$$\Theta^\alpha \stackrel{\text{def}}{=} \partial\!\!\!/\Lambda^\alpha \ , \tag{4.3}$$

defined in the same way the kernels Λ^α were defined in [12]. These are the Riesz kernels, as introduced in Subsection 1.1, with $\alpha_0 = -n+1$ and $\kappa = 1$, such that the restriction of α to the integers gives

$$\mathcal{R}^{-n-1+\alpha} = \begin{cases} \Theta^{-n+\alpha} & \text{if } \alpha \text{ is odd} \\ \Lambda^{-n-1+\alpha} & \text{if } \alpha \text{ is even.} \end{cases} \tag{4.4}$$

It is easy to see that

$$\Theta^\alpha = \frac{\Lambda^{\alpha-2}}{\alpha+n-1}\Gamma \ , \tag{4.5}$$

$$\partial_\mu\Lambda^\alpha = \frac{x_\mu - y_\mu}{\alpha+n-1}\Lambda^{\alpha-2} \ , \tag{4.6}$$

$$\gamma^\mu\Theta^\alpha = \frac{2}{\alpha+n-1}\Lambda^{\alpha-2} - \Theta^\alpha\gamma^\mu \ , \tag{4.7}$$

for $\alpha \neq -n+1$, with $\Gamma \stackrel{\text{def}}{=} \gamma^\mu(x_\mu - y_\mu)$.

Remark 4.1. *The free Dirac operator in dimension n obeys Huygens' principle if, and only if, n is odd.*

We look for a solution Ψ to (1.5) that takes the form of a series expansion such as

$$\Psi(x;y)=\sum_{k=0}^{\infty}\left\{\Theta^{\alpha_0+2k}(x-y)s_{2k}(x,y)+\Lambda^{\alpha_0+2k}(x-y)s_{2k+1}(x,y)\right\}\ , \tag{4.8}$$

where $s_k=s_k(x,y)$ is a smooth matrix coefficient and $\alpha_0=-n+1$.

We will be concerned with fields of the form $v=aI+\bar{a}\bar{\gamma}$ and $v^*=aI-\bar{a}\bar{\gamma}$. We shall assume that a and $\bar{a}$ are independent analytic functions[2].

Remark 4.2. *Consider a general* 2×2 *matrix* $A=A(x^0)$*. Then, we write*

$$A\otimes I'=aI+\bar{a}\bar{\gamma}+\theta\gamma^0+\bar{\theta}\gamma^0\bar{\gamma}\ ,$$

where $\theta=\theta(x^0)$ *and* $\bar{\theta}=\bar{\theta}(x^0)$ *are scalar functions and* I' *is the identity matrix with rank half of that of the Clifford algebra under consideration. From the equivalence*

$$e^{-\int_{y^0}^{x^0}\theta(x')dx'I}e^{\int_{y^0}^{x^0}\bar{\theta}(x')x'\bar{\gamma}}\left(\partial\!\!\!/+aI+\bar{a}\bar{\gamma}\right)e^{\int_{y^0}^{x^0}\bar{\theta}(x')dx'\bar{\gamma}}e^{\int_{y^0}^{x^0}\theta(x')dx'I}=\partial\!\!\!/+aI+\bar{a}\bar{\gamma}+\theta\gamma^0+\bar{\theta}\bar{\gamma}^0\ ,$$

we see that the case treated above is the most general effective 2×2 *perturbation of the free Dirac operator.*

Applying $\partial\!\!\!/+v$ to Ψ we find

$$\begin{aligned}(\partial\!\!\!/+v)\Psi&=\Lambda^{\alpha_0-2}\Big[s_0+\frac{2}{\alpha_0+n-1}(x^\mu-y^\mu)\partial_\mu s_0\Big]\\&+\sum_{k=0}^{\infty}\Theta^{\alpha_0+2k}\Big[-\partial\!\!\!/ s_{2k}+s_{2k+1}+v^*s_{2k}\Big]\\&+\sum_{k=1}^{\infty}\Lambda^{\alpha_0+2k-2}\Big[s_{2k}+\frac{2}{\alpha_0+2k+n-1}(x^\mu-y^\mu)\partial_\mu s_{2k}+\partial\!\!\!/ s_{2k-1}+vs_{2k-1}\Big]\ .\end{aligned}$$

Equating $(\partial\!\!\!/+v)\Psi=\Lambda^{-n-1}$ we find Hadamard's recursion:

$$\begin{aligned}s_0&=I\ ,\\s_{2k+1}&=(\partial\!\!\!/-v^*)s_{2k}\ ,\\s_{2k}+\frac{1}{k}(x^\mu-y^\mu)\partial_\mu s_{2k}&=-(\partial\!\!\!/+v)s_{2k-1}\ .\end{aligned}\tag{4.9}$$

This leads to a way of constructing explicit examples of Huygens potentials for Dirac operators by making use of the following [8]:

Theorem 4.1. *Let* $n\geq d\in\mathbb{N}$ *be an odd integer. If the Hadamard's coefficients* s_k *for the operator* $\partial\!\!\!/+v$ *vanish for all* $k\geq d$*, then* $\partial\!\!\!/+v$ *is a Huygens' type operator.*

---

[2]We stress that a,$\bar{a}$ and v,v^* do not refer to complex conjugation operations.

Notice that under the assumption that u depends only on x^0, and the fact that the Hadamard's coefficients must be bounded when $y \to x$ one can show by induction that the quantity $s_k(x, y)$ only depends on (x^0, y^0) and thus (4.9) can be re-written as

$$\begin{aligned} s_0 &= I \ , \\ s_{2k+1} &= \gamma^0 \partial_0 s_{2k} - v^* s_{2k} \ , \\ s_{2k} + \frac{1}{k}(x^0 - y^0)\partial_0 s_{2k} &= -\gamma^0 \partial_0 s_{2k-1} - v s_{2k-1} \ . \end{aligned} \tag{4.10}$$

We solve the first terms to get

$$s_0(x^0, y^0) = I \ , \tag{4.11}$$

$$s_1(x^0, y^0) = -a(x^0)I + \bar{a}(x^0)\bar{\gamma} \ , \tag{4.12}$$

$$\begin{aligned} s_2(x^0, y^0) = & \frac{1}{x^0 - y^0} \int_{x^0}^{y^0} (a(z)^2 - \bar{a}(z)^2) dz I + \frac{a(x^0) - a(y^0)}{x^0 - y^0}\gamma^0 - \\ & \frac{\bar{a}(x^0) - \bar{a}(y^0)}{x^0 - y^0}\gamma^0\bar{\gamma} \ , \end{aligned} \tag{4.13}$$

$$\begin{aligned} & s_3(x^0, y^0) = \\ & \left(\frac{a'(x^0)}{x^0 - y^0} - \frac{a(x^0) - a(y^0)}{(x^0 - y^0)^2} - \frac{a(x^0)}{x^0 - y^0} \int_{y^0}^{x^0} (a(z)^2 - \bar{a}(z)^2) dz \right) I + \\ & \left(-\frac{1}{(x^0 - y^0)^2} \int_{x^0}^{y^0} (a(z)^2 - \bar{a}(z)^2) dz + \frac{a(x^0)a(y^0)}{x^0 - y^0} - \frac{\bar{a}(x^0)\bar{a}(y^0)}{x^0 - y^0} \right) \gamma^0 + \\ & \left(-\frac{\bar{a}'(x^0)}{x^0 - y^0} + \frac{\bar{a}(x^0) - \bar{a}(y^0)}{(x^0 - y^0)^2} + \frac{\bar{a}(x^0)}{x^0 - y^0} \int_{y^0}^{x^0} (a(z)^2 - \bar{a}(z)^2) dz \right) \bar{\gamma} + \\ & \frac{a(x^0)\bar{a}(y^0) - a(y^0)\bar{a}(x^0)}{x^0 - y^0}\gamma^0\bar{\gamma} \ . \end{aligned} \tag{4.14}$$

As in the previous case of the wave operator, we can generate differential polynomials associated with the diagonal terms of the Hadamard's recursion.

Definition 4.1. *We define the diagonal terms of the Hadamard's coefficients:*

$$p_k(x^0) = s_k(x^0, x^0) \ .$$

In the present case, the polynomials will be related, as we could expect, to the AKNS hierarchy. More precisely, we find that

$$p_0 = I \ , \tag{4.15}$$

$$p_1 = -aI + \bar{a}\bar{\gamma} \ , \tag{4.16}$$

$$p_2 = (a^2 - \bar{a}^2)I + a'\gamma^0 - \bar{a}'\gamma^0\bar{\gamma} \ , \tag{4.17}$$

$$p_3 = \left(\frac{a''}{2} - a^3 + a\bar{a}^2 \right) I - \left(\frac{\bar{a}''}{2} + \bar{a}^3 - a^2\bar{a} \right) \bar{\gamma} - (a\bar{a}' - a'\bar{a})\gamma^0\bar{\gamma} \ . \tag{4.18}$$

Definition 4.2. *We define the diagonal terms of the Hadamard's coefficients, complementing Definition 4.1*

$$p_k^X(x^0) = s_k^X(x^0, x^0) \ ,$$

with $X = A, B, C, D$, where $s_k^{A,B,C,D}$ are the projection of the matrix coefficient s_k in the direction of the elements of the base of the Clifford algebra I, γ^0, $\bar{\gamma}$ and $\gamma^0\bar{\gamma}$, respectively.

Our main result reads as follows:

Theorem 4.2. *Let q, r be analytic functions related to a and $\bar{a}$ by*

$$a = -\frac{q+r}{2} \ , \quad \bar{a} = \frac{q-r}{2} \ .$$

Then, the sequence $(e_k, f_k, h_k)_{k=0}^{\infty}$ given by

$$\begin{aligned} c_k e_{2k+1} &\stackrel{def}{=} p_{2k+1}^A + p_{2k+1}^C \ , \\ c_k e_{2k} &\stackrel{def}{=} p_{2k}^B + p_{2k}^D \ , \\ c_k f_{2k+1} &\stackrel{def}{=} p_{2k+1}^A - p_{2k+1}^C \ , \\ -c_k f_{2k} &\stackrel{def}{=} p_{2k}^B - p_{2k}^D \ , \\ -c_k h_{2k+1} &\stackrel{def}{=} p_{2k+1}^D \ , \\ c_k h_{2k} &\stackrel{def}{=} p_{2k}^A \ , \end{aligned} \tag{4.19}$$

where

$$c_{k+1} = -\frac{2(k+1)}{2k+1} c_k \ , \quad c_0 = 1 \ , \tag{4.20}$$

solves the canonical AKNS recursion (2.1–2.3).

Since the proof of this result is rather long, we start off by outlining it. First we shall prove that all terms in the Hadamard's series are linear combinations of the matrices I, γ^0, $\bar{\gamma}$ and $\gamma^0\bar{\gamma}$ (Lemma 4.1). This allows breaking down the recursion into four different scalar recursions. Then comes Lemma 4.2, which is crucial in our proof. It will use Lemma A.1 (proved in the Appendix) in order to show symmetries of the Hadamard coefficients under the exchange of x and y. These symmetries are fundamental to understand the relations between

$$\partial_x^i p_k[u] = \partial_x^i \lim_{y \to x} s_k[u](x, y)$$

and

$$\lim_{y \to x} \partial_x^j s_k[u](x, y) \ ,$$

for $i, j = 0, 1, 2, \cdots$. Then, in Lemma 4.3 and Corollary 4.1, we construct a recursion for the diagonal coefficients, using the symmetries of Lemma 4.2. We also prove that one of these coefficients is always identically zero, and this simplifies our problem to three different differential recursions. This is the same number as in the AKNS recursions (2.1–2.3) or (2.10–2.12). Furthermore, in Corollary 4.1 and Remark 4.3, the recursion is equivalent to the AKNS recursion (2.10–2.12). In Lemma 4.4 we prove uniqueness of solutions of this recursion under certain conditions at infinity and prove that these conditions are satisfied by the diagonals of the Hadamard's coefficients (Lemma 4.5). Finally, in Lemma 4.6 we prove that the diagonals of the Hadamard's coefficients are isobaric differential polynomials in v and v^*. The proof is then completed by showing that the coefficients in (4.19) are isobaric solutions to the AKNS recursion.

Lemma 4.1. *Each term in the Hadamard's series can be written as*

$$s_k = s_k^A I + s_k^B \gamma^0 + s_k^C \bar{\gamma} + s_k^D \gamma^0 \bar{\gamma} \ ,$$

where $s_k^X = s_k^X(x^0)$*,* $X = A, B, C, D$*, are complex functions.*

Proof. This is a simple consequence of the fact that $\{I, \gamma^0, \bar{\gamma}, \gamma^0\bar{\gamma}\}$ is the basis of the algebra generated by itself. □

The Recursion (4.10) can then be written:

$$\begin{aligned} s_{2k+1}^{A,C} &= \partial_0 s_{2k}^{B,D} - a s_{2k}^{A,C} + \bar{a} s_{2k}^{C,A} \ , \\ s_{2k+1}^{B,D} &= \partial_0 s_{2k}^{A,C} - a s_{2k}^{B,D} - \bar{a} s_{2k}^{D,B} \ , \\ s_{2k}^{A,C} + \frac{1}{k}(x^0 - y^0)\partial_0 s_{2k}^{A,C} &= -\partial_0 s_{2k-1}^{B,D} - a s_{2k-1}^{A,C} - \bar{a} s_{2k-1}^{C,A} \ , \\ s_{2k}^{B,D} + \frac{1}{k}(x^0 - y^0)\partial_0 s_{2k}^{B,D} &= -\partial_0 s_{2k-1}^{A,C} - a s_{2k-1}^{B,D} + \bar{a} s_{2k-1}^{D,B} \ . \end{aligned} \tag{4.21}$$

Lemma 4.2. *Consider the Hadamard's coefficients for the operator* $\not{\partial} + v$*. Then*

$$s_{2k}^{A,B,D}(x^0, y^0) = s_{2k}^{A,B,D}(y^0, x^0) \ , \tag{4.22}$$

$$s_{2k}^{C}(x^0, y^0) = -s_{2k}^{C}(y^0, x^0) \ , \tag{4.23}$$

$$s_{2k+1}^{A,C}(x^0, y^0) = s_{2k+1}^{A,C}(y^0, x^0) - \frac{1}{k+1} s_{2k+2}^{B,D}(x^0, y^0)(x^0 - y^0) \ , \tag{4.24}$$

$$s_{2k+1}^{B}(x^0, y^0) = -s_{2k+1}^{B}(y^0, x^0) \ , \tag{4.25}$$

$$s_{2k+1}^{D}(x^0, y^0) = s_{2k+1}^{D}(y^0, x^0) \ . \tag{4.26}$$

This proof is the core of our main theorem in this section. Once again, we first outline it. We define a new complex matrix operator in n coordinates such that its square is the Laplacian in n Euclidean space coordinates. We reduce in 1 the total dimension of the space we are working with in order to have a new differential operator such that its fundamental solution possesses the same Hadamard's coefficients of the original Dirac operator. This new operator is self-adjoint. By Lemma A.1 it follows the symmetry in x and y of the Hadamard's coefficients.

Proof. Using Equation (4.5) we write the fundamental solution of $\bar{\partial\!\!\!/} + v$ as

$$\begin{aligned}\Psi(x;y) &= \Theta^{\alpha_0} + \sum_{k=0}^{\infty} \Lambda^{\alpha_0+2k} \left(s_{2k+1} + \Gamma \frac{s_{2k+2}}{2(k+1)} \right) = \Theta^{\alpha_0} + \qquad (4.27)\\ &\sum_{k=0}^{\infty} \Lambda^{\alpha_0+2k} \left(\left(s^A_{2k+1} + \frac{x^0-y^0}{2(k+1)} s^B_{2k+2} \right) I + \left(s^B_{2k+1} + \frac{x^0-y^0}{2(k+1)} s^A_{2k+2} \right) \gamma^0 + \right.\\ &\left(s^C_{2k+1} + \frac{x^0-y^0}{2(k+1)} s^D_{2k+2} \right) \bar{\gamma} + \left(s^D_{2k+1} + \frac{x^0-y^0}{2(k+1)} s^C_{2k+2} \right) \gamma^0 \bar{\gamma} + \\ &\left. \frac{x_j - y_j}{2(k+1)} \left(s^A_{2k+2}\gamma^j - s^B_{2k+2}\gamma^0\gamma^j - s^C_{2k+2}\gamma^j\bar{\gamma} - s^D_{2k+2}\gamma^0\gamma^j\bar{\gamma} \right) \right) , \ \alpha_0 = -n+1.\end{aligned}$$

We define the following operator:

$$\not{\mathcal{D}} = \gamma^0 \partial_0 + i \sum_{j=1}^{n'} \gamma^j \partial_j \ , \ n' = n-1 \ ,$$

We immediately see that

$$\not{\mathcal{D}}^2 = \sum_{j=0}^{n'} \partial_j^2 \stackrel{\text{def}}{=} \Delta_n \ . \qquad (4.28)$$

We write the Riesz kernels for $\not{\mathcal{D}}$ as modifications of the ones for $\bar{\partial\!\!\!/}$, i.e,

$$\begin{aligned}\widetilde{\Lambda}^\alpha(x,y) &= \widetilde{N}(\alpha) ||x-y||^\alpha \ , \qquad (4.29)\\ \widetilde{\Theta}^\alpha &= \not{\mathcal{D}} \widetilde{\Lambda}^\alpha \ , \qquad (4.30)\end{aligned}$$

where

$$\begin{aligned}||x|| \stackrel{\text{def}}{=} ||x||_{\not{\mathcal{D}}} &= \left[\sum_{j=0}^{n'} (x^j)^2 \right]^{1/2} \ ,\\ \widetilde{N}(\alpha) &= \frac{(-1)^{n'/2}}{2^{\alpha+n'+1}\pi^{(n'-1)/2}} \frac{1}{\Gamma\left(\frac{\alpha+2}{2}\right)\Gamma\left(\frac{\alpha+n'+1}{2}\right)} \ .\end{aligned}$$

These kernels have the properties

$$\widetilde{\Lambda}^{\alpha-2} = \not{\mathcal{D}} \widetilde{\Theta}^\alpha = \Delta_n \widetilde{\Lambda}^\alpha \ . \qquad (4.31)$$

This can be easily seen from Equation (4.28) and from the fact that

$$\Delta_n \widetilde{\Lambda}^\alpha = \widetilde{N}(\alpha) \left(\partial_x^2 + \frac{n-1}{x} \partial_x \right) ||x||^\alpha = \widetilde{N}(\alpha)\alpha(\alpha+n-2)||x||^{\alpha-2} = \widetilde{\Lambda}^{\alpha-2} \ ,$$

where we used that

$$\widetilde{N}(\alpha) = (\alpha+2)(\alpha+n)\widetilde{N}(\alpha+2) = (\alpha+2)(\alpha+n'+1)\widetilde{N}(\alpha+2) \ . \qquad (4.32)$$

This equation is similar to Equation (3.3) with n' instead of n. This is the reason to ignore the variable x^n, which is immaterial for the Hadamard's coefficients, as they depend only on x^0 and y^0.

We define

$$\widetilde{\Psi}(x,y) = \sum_{k=0}^{\infty} \left(\widetilde{\Theta}^{\alpha_0'+2k}(x-y) s_{2k}(x^0,y^0) + \widetilde{\Lambda}^{\alpha_0'+2k}(x-y) s_{2k+1}(x^0,y^0) \right) , \tag{4.33}$$

with $s_k(x^0,y^0)$ the Hadamard's coefficients of the operator $\not{\partial} + v(x^0)$ and $\alpha_0' = -n'+1$.

The fundamental solution of the operator $\Delta_n = \sum_{j=0}^{n'} \partial_j^2$ is given by

$$-\frac{\Gamma\left((n'+1)/2\right)}{2(n'-1)\pi^{(n'+1)/2}} \frac{1}{||x-y||^{n'-1}} = -\frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}} \frac{1}{||x-y||^{n-2}} .$$

Using the fact that $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ and that n is odd, we have

$$\widetilde{N}(-n+2) = \frac{(-1)^{(n-1)/2}}{2^2 \pi^{\frac{n}{2}-1}} \frac{1}{\Gamma\left(\frac{4-n}{2}\right)\Gamma(1)} = -\frac{(-1)^{(n-1)/2}\Gamma(n/2)\sin(\pi n/2)}{2\pi^{n/2}(n-2)} = -\frac{\Gamma(n/2)}{2\pi^{n/2}(n-2)} .$$

Thus, $\widetilde{\Lambda}^{-n+2} = \widetilde{\Lambda}^{-n'+1}$ is the fundamental solution of Δ_n.

We see that the modified kernels have the properties

$$\widetilde{\Theta}^{\alpha} = \frac{\widetilde{\Lambda}^{\alpha-2}}{\alpha+n'-1} \widetilde{\Gamma} , \tag{4.34}$$

$$\partial_\mu \widetilde{\Lambda}^{\alpha} = \frac{x_\mu - y_\mu}{\alpha+n'-1} \widetilde{\Lambda}^{\alpha-2} , \tag{4.35}$$

$$\gamma^\mu \widetilde{\Theta}^{\alpha} = \frac{2}{\alpha+n'-1} \widetilde{\Lambda}^{\alpha-2} - \widetilde{\Theta}^{\alpha}\gamma^\mu , \tag{4.36}$$

for $\alpha \neq -n'+1$, with

$$\widetilde{\Gamma} \stackrel{\text{def}}{=} \gamma^0(x_0-y_0) + i\sum_{j=1}^{n'} \gamma^j(x_j-y_j) .$$

Comparing these with Properties (4.5–4.7), we see that they are preserved if we change n to n'. Furthermore $v\widetilde{\Theta} = \widetilde{\Theta}v^*$.

Let $\mathcal{R}^\alpha$ be the Riesz kernels for $\not{\mathcal{D}}$, where $\alpha_0 = -n'+1$ and $\kappa = 1$. Then,

$$\mathcal{R}^{-n'-1+\alpha} = \begin{cases} \widetilde{\Theta}^{-n'+\alpha} \text{ if } \alpha \text{ is odd} , \\ \widetilde{\Lambda}^{-n'-1+\alpha} \text{ if } \alpha \text{ is even} . \end{cases} \tag{4.37}$$

Using the above remark and the Hadamard's recursion for the coefficients s_k, we conclude that

$$(\not{\mathcal{D}} + v(x^0))\widetilde{\Psi}(\cdot,y) = \delta_y .$$

We define

$$\mathcal{L} \stackrel{\text{def}}{=} \bar{\gamma}\left(\not{\mathcal{D}} + v(x^0)\right) = \bar{\gamma}\gamma^0\partial_0 + i\sum_{j=1}^{n'} \bar{\gamma}\gamma^j\partial_j + a(x^0)\bar{\gamma} + \bar{a}(x^0)I$$

and conclude that

$$\Phi = \widetilde{\Psi}\bar{\gamma}$$

is the fundamental solution of $\mathcal{L}$.

We also state the following properties of products of Dirac matrices (as simple consequences of Equation (4.1))

$$\begin{aligned}(\gamma^0\gamma^j)^\dagger &= \gamma^0\gamma^j \ , \\ (\gamma^0\bar{\gamma})^\dagger &= -\gamma^0\bar{\gamma} \ , \\ (\gamma^j\bar{\gamma})^\dagger &= \gamma^j\bar{\gamma} \ , \\ (\gamma^0\gamma^j\bar{\gamma})^\dagger &= \gamma^0\gamma^j\bar{\gamma} \ .\end{aligned} \tag{4.38}$$

From the properties of the gamma matrices, it is immediate that $\mathcal{L}$ is self-adjoint. This implies, from Lemma A.1, a hermitian type symmetry for Φ, namely,

$$\Phi(x;y) = \Phi(y;x)^\dagger \ . \tag{4.39}$$

We re-write the fundamental solution of $\mathcal{L}$ using Equation (4.27) as

$$\begin{aligned}\Phi(x;y) &= \widetilde{\Theta}^{\alpha_0'}\bar{\gamma} + \sum_{k=0}^{\infty}\widetilde{\Lambda}^{\alpha_0'+2k}\left(s_{2k+1}(x^0,y^0) + \widetilde{\Gamma}\frac{s_{2k+2}(x^0,y^0)}{2(k+1)}\right)\bar{\gamma} = \widetilde{\Theta}^{\alpha_0'}\bar{\gamma} + \\ &\sum_{k=0}^{\infty}\widetilde{\Lambda}^{\alpha_0'+2k}\Bigg[\left(s^A_{2k+1}(x^0,y^0) + \frac{x^0-y^0}{2(k+1)}s^B_{2k+2}(x^0,y^0)\right)\bar{\gamma} + \\ &\left(s^B_{2k+1}(x^0,y^0) + \frac{x^0-y^0}{2(k+1)}s^A_{2k+2}(x^0,y^0)\right)\gamma^0\bar{\gamma} + \\ &\left(s^C_{2k+1}(x^0,y^0) + \frac{x^0-y^0}{2(k+1)}s^D_{2k+2}(x^0,y^0)\right)I + \\ &\left(s^D_{2k+1}(x^0,y^0) + \frac{x^0-y^0}{2(k+1)}s^C_{2k+2}(x^0,y^0)\right)\gamma^0 + \\ &i\sum_{j=0}^{n'}\frac{x_j-y_j}{2(k+1)}\left(s^A_{2k+2}(x^0,y^0)\gamma^j\bar{\gamma} - s^B_{2k+2}(x^0,y^0)\gamma^0\gamma^j\bar{\gamma} - \right. \\ &\left. s^C_{2k+2}(x^0,y^0)\gamma^j - s^D_{2k+2}(x^0,y^0)\gamma^0\gamma^j\right)\Bigg] \ , \quad \alpha_0' = -n'+1 \ .\end{aligned}$$

Using the linear independence of the Clifford algebra generators and the independence of the family

$$\{\widetilde{\Lambda}^\beta(x,y) = \widetilde{\Lambda}^\beta(y,x)\}_{\beta\in\alpha_0+2\mathbb{N}} \ ,$$

it follows from (4.39) that (4.22–4.23) hold.

Furthermore, we have that

$$s^{A,C,D}_{2k+1} + \frac{x^0-y^0}{2(k+1)}s^{B,D,C}_{2k+2}$$

are symmetric under inversion of coordinates, while

$$s^B_{2k+1} + \frac{x^0 - y^0}{2(k+1)} s^A_{2k+2}$$

is anti-symmetric. This implies, in turn, Equation (4.24). For (4.25) ((4.26)), we use that s^A_{2k+2} (s^C_{2k+2}) is symmetric (anti-symmetric). □

Lemma 4.3. *The diagonal entries of the Hadamard's coefficients p_k satisfy*

$$\begin{aligned}
\partial_0 p^{A,B,D}_{2k}(x^0) &= 2 \lim_{y^0 \to x^0} \partial_0 s^{A,B,D}_{2k}(x^0, y^0) \ , & (4.40)\\
p^C_{2k}(x^0) &= 0 \ , & (4.41)\\
\partial_0 p^A_{2k+1}(x^0) &= \frac{2k+1}{k+1} \lim_{y^0 \to x^0} \partial_0 s^A_{2k+1}(x^0, y^0) + \frac{1}{k+1} \bar{a}(x^0) p^D_{2k+1}(x^0) \ , & (4.42)\\
p^B_{2k+1}(x^0) &= 0 \ , & (4.43)\\
\partial_0 p^C_{2k+1}(x^0) &= \frac{2k+1}{k+1} \lim_{y^0 \to x^0} \partial_0 s^C_{2k+1}(x^0, y^0) - \frac{1}{k+1} a(x^0) p^D_{2k+1}(x^0) \ , & (4.44)\\
\partial_0 p^D_{2k+1}(x^0) &= 2 \lim_{y^0 \to x^0} \partial_0 s^D_{2k+1}(x^0, y^0) \ . & (4.45)
\end{aligned}$$

Proof. Formulas (4.40), (4.41), (4.43) and (4.45) are simple consequences of Lemma 4.2. From the same lemma, we write

$$\partial_1 s^A_{2k+1}(x^0, y^0) = \partial_2 s^A_{2k+1}(y^0, x^0) - \frac{1}{k+1} s^B_{2k+2}(x^0, y^0) - \frac{1}{k+1} \partial_1 s^B_{2k+2}(x^0, y^0)(x^0 - y^0) \ ,$$

where ∂_i denotes differentiation with respect to the i-th variable and *not* with respect to x^i. This implies

$$\begin{aligned}
\partial_0 \left[s^A_{2k+1}(x^0, x^0) \right] &= \lim_{y^0 \to x^0} \left[\partial_1 s^A_{2k+1}(x^0, y^0) + \partial_2 s^A_{2k+1}(x^0, y^0) \right] = & (4.46)\\
& 2 \lim_{y^0 \to x^0} \partial_0 s^A_{2k+1}(x^0, y^0) + \frac{1}{k+1} s^B_{2k+2}(x^0, x^0) \ .
\end{aligned}$$

From Hadamard's recursion (4.21)

$$s^B_{2k+2}(x^0, x^0) = - \lim_{y^0 \to x^0} \partial_0 s^A_{2k+1}(x^0, y^0) + \bar{a}(x^0) s^D_{2k+1}(x^0, x^0) \ ,$$

and (4.46) we conclude Equation (4.42). Equation (4.44) is obtained in the same way. □

Corollary 4.1. *The diagonal entries $p^{A,B,D}_{2k}$ and $p^{A,C,D}_{2k+1}$ obey the following recursion:*

$$\begin{aligned}
\partial_x p^A_{2k} &= 2(a p^B_{2k} + \bar{a} p^D_{2k}) \ , & (4.47)\\
p^B_{2k} &= -\frac{k}{2k-1} \partial_x p^A_{2k-1} + \frac{2k}{2k-1} \bar{a} p^D_{2k-1} \ , & (4.48)\\
p^D_{2k} &= -\frac{k}{2k-1} \partial_x p^C_{2k-1} - \frac{2k}{2k-1} a p^D_{2k-1} \ , & (4.49)\\
p^A_{2k+1} &= \frac{1}{2} \partial_x p^B_{2k} - a p^A_{2k} \ , & (4.50)\\
p^C_{2k+1} &= \frac{1}{2} \partial_x p^D_{2k} + \bar{a} p^A_{2k} \ , & (4.51)\\
\partial_x p^D_{2k+1} &= -2(a p^C_{2k+1} + \bar{a} p^A_{2k+1}) \ , & (4.52)
\end{aligned}$$

starting from $p_0^A = 1$ *and* $p_0^{B,C,D} = 0$.

Proof. This is a simple re-statement of Hadamard's recursion (4.21) at $y^0 = x^0$ using Lemma 4.3. $\square$

Remark 4.3. *Note that modulo constants and re-arrangements of the terms the recursion (4.47–4.52) and the AKNS recursion (2.10–2.12) are the same.*

We proved that the diagonal of the Hadamard's coefficients solve (modulo constants and re-arrangements) the AKNS hierarchy (2.10–2.12). Now, we show that these coefficients are the unique solution of the recursion (given certain boundary conditions). We also prove in the sequel that $(p_k[u])_{k=0}^{\infty}$ is the unique isobaric differential polynomial sequence in v and v^* that solve this hierarchy. Together with the fact that the solution of the canonical AKNS hierarchy (2.1–2.3) is the unique isobaric solution of the AKNS hierarchy (2.10–2.12) (see Remark 2.1), we conclude that $p_k[u]$ is the canonical AKNS hierarchy.

Lemma 4.4. *The sequence* $(\bar{p}_k = (\bar{p}_k^A, \cdots, \bar{p}_k^D))_{k=0}^{\infty}$ *satisfying Recursion (4.47–4.52), with* $\bar{p}_0^A = 1$ *and* $\bar{p}_0^{B,C,D} = 0$*, is uniquely determined by the requirement that* $\bar{p}_1^D = 0$ *and*

$$\lim_{|x|\to\infty} \bar{p}_k^X(x) = 0 \ ,$$

$k \geq 2$*,* $X = A, B, C, D$*.*

Proof. For Equations (4.48–4.51) this is immediate. From Equations (4.47) and (4.52), $\bar{p}_{2k}^A$ and $\bar{p}_{2k+1}^D$ are defined modulo a constant, which can be obtained from the decay condition at infinity. $\square$

Lemma 4.5. *Assume that the Dirac operator* $\not{\partial} + a(x^0)I + \bar{a}(x^0)\bar{\gamma}$ *has bounded potentials* a *and* $\bar{a}$ *such that*

$$\lim_{|x|\to\infty} \left(a(x)^2 - \bar{a}(x)^2\right) = 0 \ ,$$
$$\lim_{|x|\to\infty} a^{(i)}(x) = \lim_{|x|\to\infty} \bar{a}^{(i)}(x) = 0 \ , \quad i \geq 1 \ .$$

Then, $p_1^B = p_1^D = 0$ *and*

$$\lim_{|x|\to\infty} p_k^X(x) = 0 \ , \ \forall k \geq 2.$$

Proof. From (4.12), $p_1^B = p_1^D = 0$. Now, we prove that

$$\lim_{|x^0|\to\infty} \lim_{y^0\to x^0} \partial_0^i s_{2k}(x^0, y^0) = 0 \ , \ i \geq 1, k \geq 0 \ , \tag{4.53}$$
$$\lim_{|x^0|\to\infty} \lim_{y^0\to x^0} s_{2k}(x^0, y^0) = 0 \ , k \geq 1. \tag{4.54}$$

For $k = 0$ in (4.53) and $k = 1$ in (4.54) the result follows by inspection. See Equations (4.15–4.17). Now, we show (4.53) for $k = 1$. First note that

$$s_2 + (x^0 - y^0)\partial_0 s_2 = \gamma^0(v^*)' + vv^* = \gamma^0 a'(x^0) - \gamma^0\bar{\gamma}\bar{a}'(x^0) + a(x^0)^2 - \bar{a}(x^0)^2 \ .$$

Differentiating p times and taking the limit $y^0 \to x^0$ followed by the limit $|x^0| \to \infty$, we get

$$(1+p) \lim_{|x^0|\to\infty} \lim_{y^0\to x^0} \partial_0^p s_2(x^0,y^0) = \lim_{|x^0|\to\infty} \partial_0^p(\gamma^0 a'(x^0) - \gamma^0\bar{\gamma} a'(x^0) + (a(x^0)^2 - \bar{a}(x^0)^2)) = 0 \,.$$

Thus proving (4.53) for $k = 0, 1$ and (4.54) for $k = 1$. For $k \geq 1$, we write

$$s_{2k} + \frac{1}{k}(x^0 - y^0)\partial_0 s_{2k} = -(\Box - \gamma^0(v^*)' - vv^*)s_{2k-2} \,.$$

Again, differentiating p times and taking the limit $y^0 \to x^0$ followed by the limit $|x^0| \to \infty$ we get

$$\frac{k+p}{k} \lim_{|x^0|\to\infty} \lim_{y^0\to x^0} \partial_0^p s_{2k}(x^0,y^0) = \tag{4.55}$$
$$- \lim_{|x^0|\to\infty} \lim_{y^0\to x^0} \left(\partial_0^{p+2} s_{2k-2}(x^0,y^0) - \partial_0^p \left[\left(\gamma^0 v'(x^0) + v(x^0)v^*(x^0)\right) s_{2k-2}(x^0,y^0)\right]\right) \,.$$

We proceed by induction assuming (4.53–4.54) for $k \leq k_0$. From the fact that

$$\lim_{|x^0|\to\infty} \partial_0^{p'} \left(\gamma^0(v^*)' + vv^*\right) = \lim_{|x^0|\to\infty} \partial_0^{p'} \left(\gamma^0 a'(x^0) - \gamma^0\bar{\gamma}\bar{a}'(x^0) + (a(x^0)^2 - \bar{a}(x^0)^2)\right) = 0 \tag{4.56}$$

for $p' \geq 0$, we conclude again Equations (4.53–4.54) for $k = k_0 + 1$. Finally, from Equation (4.54), we conclude that

$$\lim_{|x^0|\to\infty} p_{2k}^X(x^0) = 0 \,,$$

for $k \geq 1$ and $X = A, B, C, D$. The result for p_{2k+1}^X is a direct consequence of Equations (4.53–4.54), the boundedness of v^* at infinity and Hadamard's recursion (4.10). $\Box$

Lemma 4.6. *If we assign weights $i+1$ to the functions $a^{(i)}$ and $\bar{a}^{(i)}$ (or, alternatively, to $v^{(i)}$ and $(v^*)^{(i)}$), then the diagonal of the Hadamard's series p_k is an isobaric differential polynomial in a, $\bar{a}$ of weight k.*

Proof. Let

$$P_k^p(y^0) \stackrel{\text{def}}{=} \lim_{x^0\to y^0} \partial_0^p s_k(x^0,y^0) \,.$$

Then, by taking in account (4.10) and the Taylor expansion for v and v^*, we get for $y^0 \to x^0$

$$\begin{aligned}
P_0^p &= \delta_0^p \,, \\
P_{2k+1}^p &= \gamma^0 P_{2k}^{p+1} - \sum_{i=0}^{p} \binom{p}{i} (v^*)^{(i)} P_{2k}^{p-i} \,, \\
P_{2k}^p &= -\frac{k}{k+p}\left[\gamma^0 P_{2k-1}^{p+1} + \sum_{i=0}^{p} \binom{p}{i} v^{(i)} P_{2k-1}^{p-i}\right] \,.
\end{aligned}$$

A straightforward induction gives that the weight associated to P_k^p is $k+p$. From the fact that $p_k = P_k^0$, we prove the lemma. $\Box$

Proof. (of Theorem 4.2) Let us assign weight $i+1$ to $q^{(i)}$ and $r^{(i)}$. Then, $a^{(i)}$ and $\bar{a}^{(i)}$ also have weight $i+1$. From Lemma 4.6 we have that the non-zero p_k^X are differential polynomials in a and $\bar{a}$ (and, consequently in q and r) of weight k. After a simple substitution in Corollary 4.1, we find that the e_k, f_k and h_k in (4.19) solve the AKNS recursion (2.10–2.12). From the uniqueness of the isobaric solution (Remark 2.1) we conclude the proof. □

In the above theorem, at the light of Equation (4.41) and (4.43), we see that if (q,r) is a stationary solution of all the flows of the AKNS hierarchy, starting at k_0, then $p_k(x^0) = s_k(x^0,x^0) = 0$, for $k \geq k_0$. In order to relate this fact to terminating expansions and, consequently, to Huygens' type operators, we prove the following lemma:

Lemma 4.7. *Let us assume that both a and $\bar{a}$ are analytic functions*[3]*. If $s_k(x^0,x^0) = 0$ for $k \geq k_0$, $x^0 \in \mathbb{R}$, then $s_k(x^0,y^0) = 0$ for $k \geq k_0$, and $x^0, y^0 \in \mathbb{R}$.*

Proof. We differentiate i times Hadamard's recursion (4.9) and take the limit $y^0 \to x^0$. Then, by setting

$$f(x^0,y^0)|_{y^0=x^0} \stackrel{\mathrm{def}}{=} \lim_{y^0\to x^0} f(x^0,y^0) \ .$$

we find that

$$\gamma^0\partial_0^{i+1}s_{2k}(x^0,y^0)|_{y^0=x^0} = \partial_0^i s_{2k+1}(x^0,y^0)|_{y^0=x^0} + \sum_{j=0}^{i}\binom{i}{j}\partial_0^j v^*(x^0)\partial_0^{i-j}s_{2k}(x^0,y^0)|_{y^0=x^0} \ , \tag{4.57}$$

$$-\gamma^0\partial_0^{i+1}s_{2k-1}(x^0,y^0)|_{y^0=x^0} = \frac{i+k}{k}\partial_0^i s_{2k}(x^0,y^0)|_{y^0=x^0} + \sum_{j=0}^{i}\binom{i}{j}\partial_0^j v(x^0)\partial_0^{i-j}s_{2k-1}(x^0,y^0)|_{y^0=x^0} \ . \tag{4.58}$$

We proceed by induction. For $i=0$,

$$s_k(x^0,y^0)|_{y^0=x^0} = 0 \ , \ \ k \geq k_0$$

by assumption. We assume that

$$\partial_0^i s_k(x^0,y^0)|_{y^0=x^0} = 0$$

for $i \leq i_0$ and $k \geq k_0$. From Equations (4.57) and (4.58) we conclude

$$\partial_0^{i_0+1}s_k(x^0,y^0)|_{y^0=x^0} = 0 \ , \quad k \geq k_0.$$

The result follows considering the Taylor expansion in the first variable around $y^0 = x^0$ of $s_k(x^0,y^0)$, i.e.,

$$s_k(x^0,y^0) = \sum_{i=0}^{\infty}\partial_0^i s_k(x^0,y^0)|_{y^0=x^0}\frac{(x^0-y^0)^i}{i!} \ .$$

□

[3] We recall that a and $\bar{a}$ do not refer to complex conjugates of one another.

Remark 4.4. *Theorem 4.2 and its consequences were obtained under the convenient hypothesis that q and r (or, alternatively a and $\bar{a}$) are analytic functions. Moreover, the coefficients that appear in the differential polynomials p_k^X, $X = A, B, C, D$ do not depend on such assumption. In particular, it is plain that the connection between the Hadamard's hierarchy and the AKNS hierarchy remains true provided q and r are sufficiently smooth so that all terms in Equation (4.19) are defined. In other words, the hypothesis of analyticity in our result can certainly be relaxed.*

We now apply the previous result to produce examples of Dirac operators of Huygens type.

Theorem 4.3. *If a Dirac operator of the form*

$$\partial\!\!\!/ - \frac{q+r}{2} I + \frac{q-r}{2} \bar{\gamma} \ , \tag{4.59}$$

has a terminating Hadamard's expansion at $k_0 - 1$, i.e., $s_k(x^0, y^0) = 0$, $k \geq k_0$, then (q, r) is a stationary solution of the canonical AKNS hierarchy flows for $k \geq k_0$. Furthermore, if (q, r) is a stationary solution of the AKNS hierarchy (2.10–2.12), i.e., $e_k = f_k = h_k = 0$, $k \geq k_0$, with

$$\lim_{|x| \to \infty} q^{(i)} r^{(j)} = 0 \ , \quad \forall i, j \geq 0 \ , \tag{4.60}$$

then the operator given by (4.59) truncates at $k_0 - 1$.

Proof. First we suppose that the expansion is terminating. Then, we have that $s_k(x^0, y^0) = 0$, $k \geq k_0$, from Theorem 4.2 and the fact that the relation between p_k^X and e_k, f_k and h_k does not depend on analyticity, we conclude that $e_k = f_k = h_k = 0$, $\forall k \geq k_0$.

Now, take q and r such that $e_k = f_k = h_k = 0$, $k \geq k_0$ for some solution of the recursion (2.10–2.12). Consider the diagonal of the Hadamard's coefficients and define

$$\begin{aligned}
c_k \tilde{e}_{2k+1} &\stackrel{\text{def}}{=} p_{2k+1}^A + p_{2k+1}^C \ , \\
c_k \tilde{e}_{2k} &\stackrel{\text{def}}{=} p_{2k}^B + p_{2k}^D \ , \\
c_k \tilde{f}_{2k+1} &\stackrel{\text{def}}{=} p_{2k+1}^A - p_{2k+1}^C \ , \\
-c_k \tilde{f}_{2k} &\stackrel{\text{def}}{=} p_{2k}^B - p_{2k}^D \ , \\
-c_k \tilde{h}_{2k+1} &\stackrel{\text{def}}{=} p_{2k+1}^D \ , \\
c_k \tilde{h}_{2k} &\stackrel{\text{def}}{=} p_{2k}^A \ ,
\end{aligned}$$

where

$$c_{k+1} = -\frac{2(k+1)}{2k+1} c_k \ , \quad c_0 = 1 \ . \tag{4.61}$$

A simple inspection in Corollary 4.1 shows that $(\tilde{e}_k, \tilde{f}_k, \tilde{h}_k)$ is a particular solution of recursion (2.10–2.12). From Lemma 2.2 we write:

$$\begin{aligned}
\tilde{e}_k &= \sum_{i=0}^{k} d_i e_{k-i} \ , \\
\tilde{f}_k &= \sum_{i=0}^{k} d_i f_{k-i} \ , \\
\tilde{h}_k &= \sum_{i=0}^{k} d_i h_{k-i} \ ,
\end{aligned}$$

for a certain set of constants d_i, $i \geq 0$, $d_0 = 1$. We write

$$\begin{aligned}
p_{2k}^{A} &= \sum_{i=0}^{2k} d_{2k-i} h_i \ , \\
p_{2k+1}^{D} &= \sum_{i=0}^{2k+1} d_{2k+1-i} h_i \ .
\end{aligned}$$

We take the limit $|x^0| \to \infty$, and from Lemmas 2.1 and 4.5, we have that $d_0 = 1$, $d_k = 0$, $k \geq 0$. Thus,

$$\begin{aligned}
c_k e_{2k+1} &= p_{2k+1}^{A} + p_{2k+1}^{C} \ , \\
c_k e_{2k} &= p_{2k}^{B} + p_{2k}^{D} \ , \\
c_k f_{2k+1} &= p_{2k+1}^{A} - p_{2k+1}^{C} \ , \\
-c_k f_{2k} &= p_{2k}^{B} - p_{2k}^{D} \ , \\
-c_k h_{2k+1} &= p_{2k+1}^{D} \ , \\
c_k h_{2k} &= p_{2k}^{A} \ .
\end{aligned}$$

We immediately conclude that $p_k^X = 0$, $k \geq k_0$ and from Lemma 4.7, we conclude that $s_k(x^0, y^0) = 0$, $k \geq k_0$. □

Remark 4.5. *The assumption (4.60) is not superfluous. This can be seen by the following example. Consider* $q = 1$ *and* $r = 2\wp(x^0)$, *where* $\wp(x^0)$ *is the elliptic Weierstrass-P function [17]. These functions are stationary solutions of the AKNS hierarchy (see [17, 16]) with* $k_0 = 4$ *but do not obey the decay conditions at infinity. Note that they are not solutions of the canonical AKNS hierarchy. An explicit calculation shows that the Hadamard's series of the operator (4.59) does not truncate at* $k = 3$.

Remark 4.6. *As a corollary of the proof of Theorem 4.3, it follows that if a pair* (q, r) *vanishes at infinity according to assumptions (4.60) and solves the AKNS hierarchy then it must be a solution of canonical hierarchy.*

As a direct consequence of Theorem 4.2 and Lemma 4.7 we have the following:

Corollary 4.2. *Let (q, r) be a stationary solution of the canonical AKNS hierarchy, i.e., $e_k[q, r] = f_k[q, r] = 0$, $\forall k \geq k_0$ and let $n \geq k_0$ be an odd integer. Then, the Dirac operator given by (4.59) in n space dimensions is of Huygens' type.*

5 Concluding Remarks

We conclude this article with some examples of Huygens' type operators and with a heuristical discussion that aims in bringing our main results within the framework of what we call *Hadamard's hierarchy.*

5.1 Hadamard's Hierarchy

Let us suppose that the free operator $\mathcal{L}$ is invariant by the Poincaré group. This implies that the Riesz kernels associated to it depend only on $||x - y||_{\mathcal{L}}$ (and not on x and y individually).

Now, for fixed $\mathcal{L} + u$, let us consider the diagonal of the Hadamard's coefficients, $p_k[u](x) \stackrel{\text{def}}{=} s_k(x, x)$ and define the functionals

$$\mathcal{F}_k[u](x) \stackrel{\text{def}}{=} p_k[u](x) \ .$$

From the invariance by translations $(x, y) \mapsto (x + x', y + x')$ of $\mathcal{L}$, $\mathcal{R}^\alpha(x, y)$ and $\delta_y(x)$, we conclude that

$$\mathcal{T}_{x'} \left[\mathcal{F}_k[u]\right](x) = \mathcal{F}_k[u](x + x') = p_k[u](x + x') = \mathcal{F}_k \left[\mathcal{T}_{x'}[u]\right](x) \ ,$$

where

$$\mathcal{T}_{x'}[f](x) \stackrel{\text{def}}{=} f(x + x')$$

is the translation operator. We find that

$$\mathcal{T}_{x'} \left[p_k[u]\right](x) = p_k \left[\mathcal{T}_{x'}[u]\right](x) \ ,$$

and thus

$$p_k \left[\mathcal{T}_x[u]\right](0) = p_k[u](x) \ ,$$

i.e., the coefficient p_k depends on x only through functionals of $u(x)$.

Now, let us also assume that the perturbation depends on x^0 and on some finite (but of arbitrary size) set of new variables (that will be frequently omitted) $t_1, t_2, t_3, \cdots$. For the same reasons as before we conclude that $p_k[u]$ does not depend on t_j, $j = 1, 2, \cdots$ explicitly. This implies, in particular, that $p_k[u]$ is a differential polynomial in u with constant coefficients. It is natural to identify $t_1 = x^0$.

If, for a particular choice of $\mathcal{L}$, we have the commutation property [4]

[4] If Condition (5.1) is not satisfied then the existence of $u = u(x; t_1, t_2, \cdots)$ satisfying (5.2) is not assured. To avoid such pathological situation we request (5.1) in the definition of the hierarchy.

$$\partial_{t_k} p_j[u] = \partial_{t_j} p_k[u] \ , \ \forall j, k \geq 1 \ , \tag{5.1}$$

we define the *Hadamard's hierarchy*

$$\partial_{t_k} u = \partial_x p_k[u] \ , \tag{5.2}$$

for $k \geq 1$, where we identified $x = x^0$.

At the light of the Hadamard's hierarchy concept, we can say that in this work we were concerned with the following problem:

> *Find examples of operators $\mathcal{L}$ such that the associated Hadamard's hierarchy is integrable.*

We provided two examples, namely:

- If $\mathcal{L} = \Box$, the D'Alembertian, then *the Hadamard's hierarchy is the KdV hierarchy* (Theorem 3.2);
- If $\mathcal{L} = \not\partial$, the Dirac operator, then *the Hadamard's hierarchy includes, as particular case, the AKNS hierarchy* (Theorem 4.2).

5.2 Examples of Huygens' Operators

We now explore some consequences of the above theorems at the wave operator level.

Theorem 5.1. *Let $\not\partial + v$ be a Dirac operator with terminating expansion. Then, $\Box \otimes I + u$ with*

$$u = -(\not\partial v^* + vv^*)$$

is terminating and thus of Huygens' type.

Proof. The even terms of the Hadamard's' series for the operator $\not\partial + v$ solve the recursion

$$s_{2k} + \frac{1}{k}(x^\mu - y^\mu)\partial_\mu s_{2k} = -(\Box - \not\partial v^* - vv^*)s_{2k-2} \ .$$

By assumption, there exists k_0 such that $s_k = 0$, $k \geq k_0$. On the other hand, the Hadamard's' coefficients for the operator $\Box \otimes I + u$ solve

$$w_k + \frac{1}{k}(x^\mu - y^\mu)\partial_\mu w_k = -(\Box \otimes I + u)w_{k-1} \ .$$

By the uniqueness, we find that $w_k = s_{2k}$, and then $w_k = 0$, $k \geq k_0/2$. $\Box$

Corollary 5.1. *Let q be a solution of the mKdV hierarchy such that*

$$\lim_{|x|\to\infty} q^{(i)} = 0 \ , \quad i \geq 0 \ ,$$

and let $u^\pm \stackrel{def}{=} \pm q' - q^2$ be its Miura transformations. Then $\Box - u$ is of Huygens' type.

Proof. Let us consider $q = r$ in the operator given by Equation (4.59). Then, $v = -qI$ and $u = -\gamma^0 v' - v^2 = (-q' + q^2) \oplus (q' + q^2) = -(u^+ \oplus u^-)$, in the Pauli-Dirac representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \bar{\gamma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{5.3}$$

Then, $w_k = w_k^- \oplus w_k^+$, where

$$w_k^{\pm} + \frac{1}{k}(x^\mu - y^\mu)\partial_\mu w_k^{\pm} = -(\Box - u^{\pm})w_{k-1}^{\pm} .$$

From Theorems 4.3 and 5.1 the claim follows. □

Another direct consequence of Theorems 4.3 and 5.1 is the following:

Corollary 5.2. *For (q, r) as in Theorem 4.3, the matrix-wave operator given by*

$$\Box \otimes I + \frac{q' + r'}{2}\gamma^0 + \frac{q' - r'}{2}\gamma^0\bar{\gamma} - qrI$$

is of Huygens' type.

Example 5.1. *Corollary 5.2 gives us a powerful method to obtain new examples of Huygens' type operator as matrix perturbations of the wave operator. For example, if u is a solution of the KdV hierarchy, with*

$$\lim_{|x| \to \infty} u^{(i)} = 0 , \quad i \geq 0 ,$$

then $(u, 1)$ solves the AKNS obeying condition (4.60). Thus,

$$\not{\partial} - \frac{u + 1}{2}I + \frac{u - 1}{2}\bar{\gamma}$$

is of Huygens' type, and consequently

$$\Box \otimes I + \frac{u'}{2}\gamma^0(I + \bar{\gamma}) - uI ,$$

is also in the matrix-wave operator case. For $u = 2/(x^0)^2$ the solution is given by,

$$\Psi(x; y) = \Lambda^{-n+1} + \Lambda^{-n+3}\frac{2}{(x^0)^2(y^0)^2}\left((x^0 + y^0)\frac{\gamma^0}{2}(I + \bar{\gamma}) + x^0 y^0 I\right) .$$

while for

$$u = \frac{6t(2 + t^3)}{(1 - t^3)^2}$$

the Hadamard's series can be obtained explicitly and $w_k = 0$ for $k \geq 3$.

Finally, one generalization of Miura's transformation follows naturally from our construction. In the Pauli-Dirac representation (5.3) the operator in Corollary 5.2 is

$$\Box \otimes I - \begin{pmatrix} qr - \frac{q'+r'}{2} & -\frac{q'-r'}{2} \\ \frac{q'-r'}{2} & qr + \frac{q'+r'}{2} \end{pmatrix} .$$

If we impose $q = r$ (i.e., the mKdV hierarchy) we find a decomposition of the above operator in $(\Box + q^2 - q') \oplus (\Box + q^2 + q')$. We can say, then that the transformation

$$(q, r) \mapsto \begin{pmatrix} -qr + \frac{q'+r'}{2} & \frac{q'-r'}{2} \\ -\frac{q'-r'}{2} & -qr - \frac{q'+r'}{2} \end{pmatrix}$$

generalizes of the Miura transformation $q \mapsto \pm q' - q^2$.

A Appendix: One Auxiliary Lemma

We prove a useful lemma that is needed in the proof of the main result. It states that the fundamental solution of a (formally) self-adjoint operator is hermitian. By a formally self-adjoint operator L we mean that

$$\int_{\mathcal{O}} \mathsf{L}\phi \cdot \psi^\dagger dx = \int_{\mathcal{O}} \phi \cdot (\mathsf{L}\psi)^\dagger dx \ , \quad \forall \phi, \psi \in C_0^\infty(\mathcal{O}; \mathbb{C}^m) \ . \tag{A.1}$$

Here, as usual, $\mathcal{O}$ is a nonempty open subset of $\mathbb{R}^{n+1}$.

Lemma A.1. *Let $\mathsf{L} = \mathsf{L}(x, \partial_x)$ be a formally self-adjoint operator with matrix coefficients in $C^\infty(\mathcal{O})$. Assume that $\Phi(x; y)$ is a fundamental solution of L in the sense that*

$$\mathsf{L}\Phi(x; y) = \delta_y(x) \ .$$

Then, for every pair $(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ where Φ is regular[5] *we have the hermitian property*

$$\Phi(x; y) = \Phi(y; x)^\dagger \ . \tag{A.2}$$

Proof. As usual, we endow $C_0^\infty(\mathcal{O}; \mathbb{C}^m)$ with the semi-norms

$$p_{K,\alpha}(\varphi) = \sup_{x \in K} |\partial^\alpha \phi(x)| \ ,$$

where $K \subset \mathcal{O}$ is a compact set and α is a multi-index. We denote such locally convex space of test functions by $\mathcal{D} = \mathcal{D}(\mathcal{O}; \mathbb{C}^m)$ and by its topological dual $\mathcal{D}' = \mathcal{D}'(\mathcal{O}; \mathbb{C}^m)$. The latter being the space of vector-valued distributions (c.f. page 192, [20]). The duality will be denoted by $\langle R^T, \varphi \rangle = \sum_{i=1}^m \langle R_i, \varphi_i \rangle$, where the sub-index i denotes the i-th component of the vector R (resp. φ). Equation (A.1) takes the form

$$\langle (\mathsf{L}\varphi)^\dagger, \psi \rangle = \langle \varphi^\dagger, \mathsf{L}\psi \rangle \ , \forall \varphi, \psi \in \mathcal{D}(\mathcal{O}; \mathbb{C}^m) \ . \tag{A.3}$$

[5] By regular we mean that it coincides with a continuous function in a neighborhood of the point.

We will show that in the sense of distributions (i.e., in $\mathcal{D}'(\mathcal{O} \times \mathcal{O}; \mathbb{C}^m)$) we have $\Phi(x;y) = \Phi(y;x)^\dagger$. In other words, we will show that we have

$$\langle \Phi_{ij}(x;y), \eta(x,y) \rangle = \langle \overline{\Phi_{ji}}(x;y), \eta(x,y) \rangle \ , \forall \eta \in \mathcal{D}(\mathcal{O} \times \mathcal{O}; \mathbb{C})$$

But in view of the denseness of functions of the form $\sum_l \alpha_l(x)\beta_l(y)$ in $\mathcal{D}(\mathcal{O} \times \mathcal{O}; \mathbb{C})$ for $\alpha, \beta \in \mathcal{D}(\mathcal{O}; \mathbb{C})$ it is enough to show that

$$\langle \Phi_{ij}(x;y), \alpha(x)\beta(y) \rangle = \langle \overline{\Phi_{ji}}(x;y), \alpha(y)\beta(x) \rangle \ , \forall \alpha, \beta \in \mathcal{D}(\mathcal{O}; \mathbb{C}) \ . \qquad \text{(A.4)}$$

We thus consider $\varphi, \psi \in \mathcal{D}(\mathcal{O}; \mathbb{C}^m)$, set $\widetilde{\varphi}(x) = \langle \Phi(x;y), \varphi(y) \rangle$ and $\widetilde{\psi}(x) = \langle \Phi(x;y), \psi(y) \rangle$. Notice that $\widetilde{\varphi}$ and $\widetilde{\psi}$ are in $\mathcal{D}(\mathcal{O}; \mathbb{C}^m)$ and a standard argument in $\mathcal{D}'(\mathcal{O} \times \mathcal{O}; \mathbb{C})$ gives that

$$\mathsf{L}\widetilde{\varphi} = \varphi \text{ and } \mathsf{L}\widetilde{\psi} = \psi \ .$$

Thus, using Equation (A.3) we get

$$\begin{aligned} \langle \varphi^\dagger(x), \langle \Phi(x;y), \psi(y) \rangle \rangle &= \langle \langle \Phi(x;y), \varphi(y) \rangle^\dagger, \psi(x) \rangle \ , \text{ i.e.} \\ \langle \langle \varphi(x)^\dagger \Phi(x;y), \psi(y) \rangle \rangle &= \langle \langle \varphi^\dagger(y) \Phi^\dagger(x,y), \psi(x) \rangle \rangle \ , \text{ i.e.} \\ \langle \langle (\Phi^\dagger(x;y)\varphi(x))^\dagger, \psi(y) \rangle \rangle &= \langle \langle (\Phi(x;y)\varphi(y))^\dagger, \psi(x) \rangle \rangle \ . \end{aligned}$$

To prove Equation (A.4), we now choose $\varphi(x) = \alpha(x)e_j$ and $\psi(y) = \beta(y)e_i$, where e_l denotes the l-th vector of the canonical basis in $\mathbb{R}^{n+1}$. $\square$

References

[1] Mark J. Ablowitz, David J. Kaup, Alan C. Newell, and Harvey Segur. The inverse scattering transform-Fourier analysis for nonlinear problems. *Studies in Appl. Math.*, 53(4):249–315, 1974.

[2] M. Adler and J. Moser. On a class of polynomials connected with the Korteweg-de Vries equation. *Comm. Math. Phys.*, 61(1):1–30, 1978.

[3] H. Airault, H. P. McKean, and J. Moser. Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem. *Comm. Pure Appl. Math.*, 30(1):95–148, 1977.

[4] Yu. Yu. Berest and A. P. Veselov. The Huygens principle and integrability. *Uspekhi Mat. Nauk*, 49(6(300)):7–78, 1994.

[5] Yuri Berest. Lacunae of hyperbolic Riesz kernels and commutative rings of partial differential operators. *Lett. Math. Phys.*, 41(3):227–235, 1997.

[6] Yuri Berest. Hierarchies of Huygens' operators and Hadamard's conjecture. *Acta Appl. Math.*, 53(2):125–185, 1998.

[7] Yuri Yu. Berest. Solution of a restricted Hadamard problem on Minkowski spaces. *Comm. Pure Appl. Math.*, 50(10):1019–1052, 1997.

[8] F. A. C. C. Chalub and J. P. Zubelli. Huygens' Principle, Dirac Operators, and Rational Solutions of the AKNS Hierarchy, 2004. To appear: Mathematical Physics, Analysis and Geometry.

[9] D. V. Choodnovsky and G. V. Choodnovsky. Pole expansion of nonlinear differential equations. *Nuovo Cimento*, 40B:339–353, 1977.

[10] J. J. Duistermaat and F. A. Grünbaum. Differential equations in the spectral parameter. *Comm. Math. Phys.*, 103(2):177–240, 1986.

[11] H. Flaschka, A. C. Newell, and T. Ratiu. Kac-Moody Lie algebras and soliton equations. II. Lax equations associated with $A_1^{(1)}$. *Phys. D*, 9(3):300–323, 1983.

[12] Gerald B. Folland. Fundamental solutions for the wave operator. *Exposition. Math.*, 15(1):25–52, 1997.

[13] F. G. Friedlander. *The wave equation on a curved space-time*. Cambridge University Press, Cambridge, 1975. Cambridge Monographs on Mathematical Physics, No. 2.

[14] I. M. Gel′fand and L. A. Dikiĭ. Fractional powers of operators, and Hamiltonian systems. *Funkcional. Anal. i Priložen.*, 10(4):13–29, 1976.

[15] I. M. Gel′fand and L. A. Dikiĭ. The resolvent, and Hamiltonian systems. *Funkcional. Anal. i Priložen.*, 11(2):11–27, 95, 1977.

[16] Fritz Gesztesy and Helge Holden. Darboux-type transformations and hyperelliptic curves. *J. Reine Angew. Math.*, 527:151–183, 2000.

[17] Fritz Gesztesy and Rudi Weikard. Elliptic algebro-geometric solutions of the KdV and AKNS hierarchies—an analytic approach. *Bull. Amer. Math. Soc. (N.S.)*, 35(4):271–317, 1998.

[18] John E. Gilbert and Margaret A. M. Murray. *Clifford algebras and Dirac operators in harmonic analysis*. Cambridge University Press, Cambridge, 1991.

[19] F. John. *Partial differential equations*. Springer-Verlag, New York, 1971. Applied Mathematical Sciences, Vol. 1.

[20] A.A. Kirillov and A.D. Gvishiani. *Theorems and Problems in Functional Analysis*. Springer Verlag, New York, Heidelberg, Berlin, 1982.

[21] J. E. Lagnese and K. L. Stellmacher. A method of generating classes of Huygens' operators. *J. Math. Mech.*, 17:461–472, 1967.

[22] N. G. Marchuk. Dirac γ-equation, classical gauge fields and Clifford algebra. *Adv. Appl. Clifford Algebras*, 8(1):181–225, 1998.

[23] Alan C. Newell. *Solitons in mathematics and physics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1985.

[24] Robert L. Sachs. On the integrable variant of the Boussinesq system: Painlevé property, rational solutions, a related many-body system, and equivalence with the AKNS hierarchy. *Phys. D*, 30(1-2):1–27, 1988.

[25] Rainer Schimming. Korteweg-de vries Hierarchie und Huygensschess Prinzip, 1986.

[26] Rainer Schimming. An explicit expression for the Korteweg-de Vries hierarchy. *Acta Appl. Math.*, 39(1-3):489–505, 1995. KdV '95 (Amsterdam, 1995).

[27] Bernd Thaller. *The Dirac equation*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.

[28] George Wilson. Bispectral commutative ordinary differential operators. *J. Reine Angew. Math.*, 442:177–204, 1993.

[29] George Wilson. Collisions of Calogero-Moser particles and an adelic Grassmannian (with an appendix by I. G. Macdonald). *Invent. Math.*, 133(1):1–41, 1998.

[30] J. P. Zubelli. Differential equations in the spectral parameter for matrix differential operators. *Physica D*, 43:269–287, 1990.

[31] J. P. Zubelli. Rational solutions of nonlinear evolution equations, vertex operators, and bispectrality. *J. Differential Equations*, 97(1):71–98, 1992.