Direction of vorticity and regularity for solutions to the evolution Navier-Stokes equations under the Navier boundary condition

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Abstract

In reference [14], by Constantin and Fefferman, a quite simple geometrical assumption on the direction of the vorticity is shown to be sufficient to guarantee the regularity of the weak solutions to the evolution Navier–Stokes equations in the whole of $\mathbf{R}^3$. Essentially, the solution is regular if the direction of the vorticity is Lipschitz continuous with respect to the space variables. In reference [10], among other side results, the authors prove that 1/2-Hőlder continuity is sufficient. In [6] the sufficient conditions for regularity obtained in [10] are replaced by simple assumptions that relate the direction and the amplitude of the vorticity. The proofs are a readaptation of those in [10].

A main open problem remains of the possibility of extending the same kind of results to boundary value problems. Here, we succeed in making this extension to the well known Navier (or slip) boundary condition. Moreover, our approach simplifies some aspects in the previous proofs.

It is worth noting that our proof may be adapted to other boundary conditions. However, the extension to the non-slip boundary condition remains open.

1 Introduction

In reference [14] Charles Fefferman and Peter Constantin open the way to the study of global regularity of solutions of the Navier-Stokes equations via simple geometrical assumptions on the direction of the vorticity, a very significant, even "visible", physical entity. The literature related to this subject is wide. We will not give here a list of papers on the subject but just refer, besides the papers quoted in the above abstract, [1], [2], [3], [15], [18], [19], [26], [27]. Before going into these types of conditions we recall some of our previous results relating vorticity to regularity of solutions of the Navier-Stokes equations. We denote by $|\cdot|_p$ the canonical norm in the Lebesgue space $L^p := L^p(\mathbf{R}^3)$, $1 \leq p \leq \infty$. $H^s := H^s(\mathbf{R}^3)$, $0 \leq s$, denotes the classical Sobolev spaces. Scalar and vector function spaces are indicated by the same symbol.

Consider the evolution 3-D Navier–Stokes equations in $\mathbf{R}^3$

$$(1.1)\qquad \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)\, u - \nu \Delta u + \nabla p = 0 \quad \text{in } \mathbf{R}^3 \times [0, T], \\ \nabla \cdot u = 0 \quad \text{in } \mathbf{R}^3 \times [0, T], \\ u(x, 0) = u_0(x) \quad \text{in } \mathbf{R}^3 . \end{cases}$$

Standard devices allow the presence of external force fields $f(x, t)$ in the right hand of the above equation as well as in (1.7).

It is well known (essentially due to Leray [22]) that given any fixed $T > 0$ there exists at least a *weak solution*

$$u \in C_w(0, T; L^2) \cap L^2(0, T; H^1),$$

of the system (1.1) in (0,T), where C_w indicates weak continuity. See, for instance, the monographs [16], [21], [24], [25], [32]. Moreover, the energy estimate

$$(1.2)\qquad \frac{1}{2}|u(t)|_2^2 + \nu \int_0^t \int_{\mathbf{R}^3} |\nabla u(x, \sigma)|^2 \, dx \, d\sigma \leq \frac{1}{2}|u_0|_2^2$$

holds for each $t \in (0, T)$.

A weak solution such that

$$(1.3)\qquad u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$$

is called a *strong solution* in $[0,\, T]$. In the following, we say that u is a strong solution in $[0,\, T)$ if u is a strong solution in $[0,\, t]$, for each $t < T$. Strong solutions are regular, unique, and exist at least for some $T^* > 0$.

It is not known whether weak solutions are unique and strong solutions are global in time. Hence many efforts have been made to obtain significant conditions that are sufficient to guarantee the regularity of weak solutions.

In the following we are interested in conditions on the vorticity of the velocity field u defined as

$$\omega(x, t) = \nabla \times u(x, t) \, .$$

In the field of analytic (not geometric) assumptions on ω it is proved in reference [5] that if

$$(1.4)\qquad \omega \in L^p(0, T; L^q) \quad \text{for } \frac{2}{p} + \frac{n}{q} \leq 2, \quad 1 \leq p \leq 2,$$

then the weak solution is regular. See also [11]. However, this type of assumptions on the vorticity have an analytical character. On the contrary, references [14] and [10] furnish significant geometrical conditions.

Define the direction of the vorticity as

$$\xi(x) = \frac{\omega(x)}{|\omega(x)|} \, ,$$

and denote by $\theta(x, y, t)$ the angle between the vorticity ω at two distinct points x and y at time t. In reference [14] Constantin and Fefferman prove that if u is a weak solution of (1.1) in $(0, T)$ with $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$ and if

$$\sin \theta(x, y, t) \leq \, c \, |x - \, y|$$

in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K, then the solution u is strong in $[0, T]$ and, consequently, is regular. Actually, the literal statement in [14] is a little different (see in particular the comment after equation (32) in the above reference). Main ingredients in the proof of the above result are Biot-Savart Law and a particularly significant formula introduced in reference [13]. See equation (7) in [14].

In [10], Berselli and the author improve the above result by showing that

$$\sin\theta(x, y, t) \leq c|x - y|^{1/2}$$

is sufficient to guarantee the regularity of weak solutions. More precisely, in [10] we prove that if u is a weak solution of (1.1) in $(0, T)$ with $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$ and if for some $\beta \in [1/2, 1]$ one has $|\sin\theta(x, y, t)| \leq g(t, x)|x - y|^\beta$ in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K, where $g \in L^a(0, T; L^b)$ and $(2/a) + (3/b) = \beta - (1/2), \quad a \in [\frac{4}{2\beta-1}, \infty]$, then u is a strong strong in $[0, T]$ and, consequently, is regular.

In [6] and [9], by following the proof given in [10], we consider some cases in which $\beta \in [0, 1/2]$ and give a sufficient condition for the regularity of weak solutions that involves, simultaneously, the modulus and the direction of the vorticity. More precisely, we prove that if u is a weak solution of (1.1) in $(0, T)$ with $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$ and if, for some $\beta \in [0, 1/2]$, $|\sin\theta(x, y, t)| \leq c|x - y|^\beta$ in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K then the solution u is strong in $[0, T]$, consequently regular, provided that $\omega \in L^2(0, T; L^r)$ where $r = \frac{3}{\beta+1}$.

It is self evident that all the above hypotheses on $\sin\theta(x, y, t)$ may be relaxed by assuming that they are satisfied merely for $|x - y| < \delta$, with an arbitrary positive constant δ. We may also replace the constants δ and K by suitable functions of space and time.

One may also write sufficient conditions for regularity involving coefficients (like g in reference [10]), the parameter β, the unknown fields ω, u, etc. In particular, the parameter β may depend on t and take values in the range $(0, 3)$ by exploiting the fact that (2.46) holds for each β in this range. Some examples in the above directions can be obtained by simple modifications of the known proofs.

The reader should note that from the proofs given in references [14], [10] and [6] if follows that in the assumptions made in these references the quantity $\sin\theta(x, y, t)$ can be replaced by

$$|(\widehat{x - y}, \xi(x))\, Det(\widehat{x - y}, \xi(y), \xi(x))|\,. \tag{1.5}$$

This is already implicit in the equation (9) and the next one in reference [14]. In the proof given in the sequel this claim follows from the substitution of the above term (1.5) by its upper bound $\sin\theta(x, y, t)$ in the right hand side of equation (2.41) in order to obtain (2.43). It follows that the assumptions on $\sin\theta(x, y, t)$ can be made directly on (1.5) as well as on any other upper bound. In particular, we may use $|\cos\psi(x, y, t)|\, \sin\phi(x, y, t)$, where $\psi(x, t)$ is the angle between $\xi(x, t)$ and $x - y$, and $\phi(x, y, t)$ is the angle between $\xi(x, y, t)$ and the plane generated by $\xi(x, t)$ and $x - y$. In this regard, it would be of real interest to exhibit some specific examples of vector fields in $\mathbf{R}^3$ such that $|\sin\theta(x,y,t)| \leq c\,|x-y|^\beta$ holds *for all* $x, y \in \mathbf{R}^3$ but $|\sin\phi(x,y,t)| \leq c\,|x-y|^\beta$ does not hold.

Central open problems are the determination of the best exponent β for which the assumption (1.9) guarantees the regularity of the solutions *without* any other additional hypotheses, and the extension of the basic theory to boundary value problems. Our aim here is to give a first contribution to the second of these problems by extending to the Navier (or slip) boundary condition (1.16) the results proved in [6]. In order to show the new points without hiding ideas and methods in a formally more general setting, we will just consider the above problem in the half-space $\mathbf{R}^3_+$.

The extension of our results to the non-slip boundary condition

$$u = 0 \quad \textit{on} \quad \Gamma \tag{1.6}$$

under the *sole* assumption $\sin\theta(x,y,t) \leq c|x-y|^{1/2}$ remains an open problem. See the remarks after (2.4).

The slip boundary condition (1.16) is an appropriate model for many important flow problems. Besides the pioneering mathematical contribution [30] by Solonnikov and Ščadilov, this boundary condition has been considered by many authors. See [7], [8] and references therein as, for instance, [4], [12], [17], [20], [23], [28], [29], [33].

Our approach provides some simplification with respect to the previous proofs. In fact, we prove the identity (2.28) by a quite direct manipulation of the non linear term $(\omega\cdot\nabla)\,u\cdot\omega$. Note that in the particular case $\Omega = \mathbf{R}^3$ the right hand side of (2.28) consists of the sole first term.

The main result in this paper is the following. Definitions concerning the slip boundary condition and the functional space V are given below.

Theorem 1.1. *Let $u_0 \in V$ and let u be a weak solution of the Navier-Stokes equations in $[0,\,T)\times\mathbf{R}^3_+$, namely,*

$$\begin{cases} \frac{\partial u}{\partial t} + (u\cdot\nabla)\,u - \nu\Delta u + \nabla p = 0 \quad \textit{in } \mathbf{R}^3_+ \times [0,T), \\ \nabla\cdot\, u = 0 \quad \textit{in } \mathbf{R}^3_+ \times [0,T), \\ u(x,0) = u_0(x) \quad \textit{in } \mathbf{R}^3_+, \end{cases} \tag{1.7}$$

endowed with the slip boundary condition

$$\begin{cases} u_3 = 0, \\ \nu \dfrac{\partial u_j}{\partial x_3} = 0, \quad 1 \leq j \leq 2. \end{cases} \tag{1.8}$$

Let $\beta \in [0, 1/2]$ and assume that

$$|\sin\theta(x,y,t)| \leq c|x-y|^\beta \tag{1.9}$$

in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K. Moreover, suppose that

$$\omega \in L^2(0,T;L^r), \tag{1.10}$$

where

$$r = \frac{3}{\beta+1}. \tag{1.11}$$

Then the solution u is strong in $[0,T]$ and, consequently, is regular.

Next we introduce the *slip boundary condition.* It is superfluous to give here the well known variational formulation of the problem considered in Theorem 1.1. We merely remark that the standard functional framework in studying the boundary condition (1.8) is

$$V = \left\{ v \in [H^1(\mathbf{R}^3_+)]^2 \times H^1_0(\mathbf{R}^3_+) : \nabla \cdot v = 0 \right\} .$$

See [7].

Even though we consider here the Navier-Stokes equations in the half-space $\mathbf{R}^3_+ = \left\{ x \in \mathbf{R}^3 : \, x_3 \, > 0 \right\}$ it is suitable to describe the slip boundary condition (1.16) in the general case of an open set Ω in $\mathbf{R}^3$. Γ denotes the boundary of Ω and $\underline{n}$ the unit external normal to Γ. We denote by

$$T = -p\, I + \nu (\nabla u + \nabla u^T)$$

the stress tensor, and set $\underline{t} = T \cdot \underline{n}$. Hence, with an obvious notation

$$T_{ik} = -\delta_{ik} p + \nu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right), \tag{1.12}$$

$$t_i = \sum_{k=1}^{n} T_{ik} n_k. \tag{1.13}$$

We also define the linear operator $\underline{\tau}$,

$$\underline{\tau}(u) = \underline{t} - (\underline{t} \cdot \underline{n})\underline{n}. \tag{1.14}$$

Hence

$$\tau_i(u) = \nu \sum_{k=1}^{n} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) n_k - 2\nu \left[\sum_{k,l=1}^{n} \frac{\partial u_l}{\partial x_k} n_k n_l \right] n_i. \tag{1.15}$$

Note that $\underline{\tau}(u)$ is tangential to the boundary and independent of the pressure p.

The slip boundary condition reads

$$\begin{cases} (u \cdot \underline{n})_{|\Gamma} = 0, \\ \\ \underline{\tau}(u)_{|\Gamma} = 0\,. \end{cases} \tag{1.16}$$

For convenience, we consider here homogeneous boundary conditions.

When $\Omega = \mathbf{R}^3_+$, the equations (1.16) have the form (1.8). See [7], Equation (2.2).

2 Proof of Theorem 1.1.

From now on we set

$$\Omega = \mathbf{R}^3_+ \quad \text{and} \quad \Gamma = \left\{ x \in \mathbf{R}^3 : \, x_3 \, = 0 \right\} .$$

For convenience, we mostly will use the Ω, Γ notation.

Since $u_0 \in H^1$, the solution is strong, hence regular, in $[0, \, \tau)$, for some $\tau > 0$. Let $\tau \leq T$ be the maximum of these values. We will show that, under this hypothesis, u is strong in $[0, \, \tau]$. Hence, by a continuation principle, u is strong in $[\tau, \, \tau + \, \varepsilon)$. This shows that $\tau \, = \, T$. Without loss of generality we assume that the solution u is regular in $[0, \, T)$ and we prove that this implies regularity in $[0, \, T]$.

By taking the curl of both sides of the first equation (1.7) we find, for each $t < T$,

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla) \, \omega - \nu \Delta \omega = \, (\omega \cdot \nabla) \, u \, , \tag{2.1}$$

in $\mathbf{R}^3_+$. Moreover, by taking the scalar product in L^2 of both sides of (2.1) with ω, we get

$$\frac{1}{2} \frac{d}{dt} |\omega|_2^2 + \nu |\nabla \omega|_2^2 = \int_\Omega (\omega \cdot \nabla) \, u \, \cdot \omega(x) \, dx. \tag{2.2}$$

Note that

$$-\nu \int_\Omega \Delta \omega \cdot \, \omega \, dx = \, \nu |\nabla \omega|_2^2 + \, \nu \int_\Gamma \frac{\partial \, \omega}{\partial \, x_3} \cdot \, \omega \, d\Gamma \tag{2.3}$$

since $\underline{n} = \, (0, 0, -1)$. Under the boundary condition (1.8) it readily follows that

$$\int_\Gamma \frac{\partial \, \omega}{\partial \, x_3} \cdot \, \omega \, d\Gamma = \int_\Gamma \frac{\partial \, \omega_3}{\partial \, x_3} \cdot \, \omega_3 \, d\Gamma = \, 0 \, .$$

However, under the non-slip boundary condition (1.6) one gets

$$\int_\Gamma \frac{\partial \, \omega}{\partial \, x_3} \cdot \, \omega \, d\Gamma = \, \frac{1}{2} \, \frac{\partial}{\partial \, x_3} \, \int_\Gamma (\omega_1^2 + \, \omega_2^2) \, d\Gamma \, . \tag{2.4}$$

If we are able to control this quantity in a suitable way, then the Theorem 1.1 applies to the non-slip boundary condition as well, as easily shown by a simple adaptation of the proofs given here.

Set, for each triad (j, k, l), $j, k, l \in \, \{1, 2, 3\}$,

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i, j, k) \text{ is an even permutation} \, , \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation} \, , \\ 0 & \text{if two indexes are equal} \, . \end{cases} \tag{2.5}$$

One has

$$(a \times b)_j \, = \, \epsilon_{jkl} \, a_k \, b_l \, , \tag{2.6}$$

and

$$(\nabla \times v)_j = \epsilon_{jkl} \frac{\partial v_l}{\partial x_k}, \tag{2.7}$$

where here, and in the sequel, the usual convention about summation of repeated indexes is assumed.

Since

$$-\Delta u = \nabla \times (\nabla \times u) - \nabla (\nabla \cdot u), \tag{2.8}$$

it follows that

$$\begin{cases} -\Delta u = \nabla \times \omega \quad \text{in} \quad \Omega; \\ \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0 \quad \text{in} \quad \Gamma, \\ u_3 = 0 \quad \text{in} \quad \Gamma, \end{cases} \tag{2.9}$$

for each t.

In the sequel

$$G(x,y) = \frac{1}{4\pi} \left(\frac{1}{|x-y|} - \frac{1}{|x-\overline{y}|} \right) \tag{2.10}$$

denotes the Green's function for the Dirichlet boundary value problem in the half space, where

$$\overline{y} = (y_1, y_2, -y_3),$$

and

$$N(x,y) = \frac{1}{4\pi} \left(\frac{1}{|x-y|} + \frac{1}{|x-\overline{y}|} \right) \tag{2.11}$$

denotes the classical Neumann's function for the half space $\mathbf{R}^3_+$.

For $j = 1, 2, 3$ we set

$$\begin{cases} a_j(x) = -\frac{1}{4\pi} \int_\Omega \epsilon_{jkl} \frac{x_k - y_k}{|x-y|^3} \omega_l(y)\, dy, \\ b_j(x) = \frac{1}{4\pi} \int_\Omega \epsilon_{jkl}\, \epsilon_k \frac{x_k - \overline{y}_k}{|x-\overline{y}|^3} \omega_l(y)\, dy, \end{cases} \tag{2.12}$$

where

$$\epsilon_1 = \epsilon_2 = 1, \; \epsilon_3 = -1.$$

In our notation we often drop the symbol t specially when it may be viewed as a parameter. One has the following result:

Lemma 2.1. *For each $x \in \Omega$*

$$\frac{\partial a_j(x)}{\partial x_i} \omega_i(x)\, \omega_j(x) = \tag{2.13}$$

$$\frac{3}{4\pi} P.V. \int_\Omega (\widehat{x-y}) \cdot \omega(x)\, Det \left((\widehat{x-y}), \omega(y), \omega(x) \right) \frac{dy}{|x-y|^3}$$

and

$$\frac{\partial\, b_j(x)}{\partial\, x_i}\,\omega_i(x)\,\omega_j(x) =$$

$$\frac{1}{4\,\pi}\int_\Omega Det\,(\omega(x),\overline{\omega}(x),\omega(y))\;\frac{dy}{|x-\overline{y}|^3} \tag{2.14}$$

$$-\,\frac{3}{4\,\pi}\int_\Omega (\widehat{x-\overline{y}})\cdot\,\omega(x)\,Det\left((\widehat{\overline{x-\overline{y}}}),\omega(y),\omega(x)\right)\;\frac{dy}{|x-\overline{y}|^3}\,.$$

Proof.

By differentiation of $a_j(x)$ with respect to x_i we show that

$$\frac{\partial\, a_j(x)}{\partial\, x_i} = -\frac{1}{4\,\pi}\,P.V.\int_\Omega \epsilon_{jkl}\,\left[\frac{\delta_{ik}}{|x-y|^3}\,-\,3\,\frac{(x_i-y_i)(x_k-y_k)}{|x-y|^5}\right]\omega_l(y)\,dy\,. \tag{2.15}$$

Straightforward calculations, left to the reader (use the combinatorial ϵ-operators), show that

$$\frac{\partial\, a_j(x)}{\partial\, x_i}\,\omega_i(x)\omega_j(x) = -\frac{3}{4\,\pi}\,P.V.\int_\Omega(\widehat{x-y})\cdot\,\omega(x)\;Det\left((\widehat{x-y}),\omega(x),\omega(y)\right)\;\frac{dy}{|x-y|^3}\,. \tag{2.16}$$

This proves (2.13).

Next we consider the b term. By differentiation of $b_j(x)$ with respect to x_i one gets

$$\frac{\partial\, b_j(x)}{\partial\, x_i} = \frac{1}{4\,\pi}\;\int_\Omega \epsilon_{jkl}\,\epsilon_k\;\left[\frac{\delta_{ik}}{|x-\overline{y}|^3}\,-\,3\,\frac{(x_i-\overline{y}_i)(x_k-\overline{y}_k)}{|x-\overline{y}|^5}\right]\omega_l(y)\,dy\,. \tag{2.17}$$

Hence,

$$\frac{\partial\, b_j(x)}{\partial\, x_i}\,\omega_i(x)\omega_j(x) = \frac{1}{4\,\pi}\;\int_\Omega \epsilon_{jil}\,\epsilon_i\,\omega_i(x)\,\omega_j(x)\,\omega_l(y)\,\frac{dy}{|x-\overline{y}|^3} \tag{2.18}$$

$$-\frac{3}{4\,\pi}\;\int_\Omega \epsilon_{jkl}\;[(x_i\,-\,\overline{y}_i)\,\omega_i(x)]\;\epsilon_k\,(x_k\,-\,\overline{y}_k)\,\omega_j(x)\,\omega_l(y)\,\frac{dy}{|x-\overline{y}|^5}\,.$$

In accordance to previous notation we set

$$\overline{\omega} = \,(\omega_1,\,\omega_2,\,-\omega_3)\,.$$

It follows that

$$\frac{\partial\, b_j(x)}{\partial\, x_i}\,\omega_i(x)\omega_j(x) = -\frac{1}{4\,\pi}\;\int_\Omega Det\,(\overline{\omega}(x),\,\omega(x),\,\omega(y))\;\frac{dy}{|x-\overline{y}|^3} \tag{2.19}$$

$$+\frac{3}{4\,\pi}\;\int_\Omega\left(\,(\widehat{x-\overline{y}})\cdot\,\omega(x)\right)\,Det\left((\widehat{\overline{x-\overline{y}}}),\omega(x),\omega(y)\right)\;\frac{dy}{|x-\overline{y}|^3}\,.$$

This proves (2.14).

For $j = 3$ it follows from (2.9) that

$$u_j(x) = \int_\Omega G(x,\, y)\, (\nabla \times \omega(y))_j \, dy\,, \tag{2.20}$$

By appealing to (2.7) and by taking into account that $G(x,\, y) = 0$ if $y \in \Gamma$, an integration by parts yields

$$u_j(x) = -\int_\Omega \epsilon_{jkl}\, \frac{\partial\, G(x,y)}{\partial\, y_k}\, \omega_l(y)\, dy\,. \tag{2.21}$$

Hence, for $j = 3$, (2.10) shows that

$$u_j(x) = a_j(x) + b_j(x)\,. \tag{2.22}$$

Remark. Note that for the boundary value problem (1.6) the equation (2.22) holds for $j = 1, 2, 3$. This easily would lead, just by simplifying the proofs presented in the sequel, to the extension of Theorem 1.1 to solutions of the boundary value problem (1.6) *provided that* one is able to control the boundary integral (2.4).

By appealing to (2.7) and (2.9) it follows that

$$\begin{cases} -\Delta\, u_j = \epsilon_{jkl}\, \frac{\partial\, \omega_l}{\partial\, x_k}\,, \quad \text{in} \quad \Omega\,, \\ \\ \frac{\partial u_j}{\partial x_3} = 0, \quad \text{in} \quad \Gamma\,, \end{cases} \tag{2.23}$$

where $j = 1$ or $j = 2$. From (2.23) one gets

$$u_j(x) = \int_\Omega N(x,y)\, \epsilon_{jkl}\, \frac{\partial\, \omega_l(y)}{\partial\, y_k}\, dy\,, \tag{2.24}$$

for $j = 1,\, 2$. Hence, for $j = 1, 2$,(ujn2)

$$u_j(x) = -\int_\Omega \epsilon_{jkl}\, \frac{\partial\, N(x,y)}{\partial\, y_k}\, \omega_l(y)\, dy \,+\, \gamma_j(x)\,, \tag{2.25}$$

where

$$\gamma_j(x) = \int_\Gamma N(x,y)\, \epsilon_{jkl}\, \omega_l(y)\, n_k\, dy \tag{2.26}$$

is defined for $j = 1, 2, 3$ and $n_k = (0, 0, -1)$. Note that $\gamma_3(x) = 0$.

It readily follows, by appealing to (2.22) when $j = 3$, that

$$u_j(x) = a_j(x) - \epsilon_j\, b_j(x) + \gamma_j(x)\,, \quad j = 1, 2, 3. \tag{2.27}$$

Remark. Under the non-slip boundary condition (2.27) is replaced by (2.22), which holds for each j. This leads to a simpler situation.

From (2.27), (2.13) and (2.14) careful, but straightforward, calculations show that

$$((\omega\cdot\nabla)\,u\cdot\,\omega)\,(x) \equiv \tfrac{\partial\, u_j(x)}{\partial\, x_i}\,\omega_i(x)\omega_j(x) =$$

$$-\tfrac{3}{4\,\pi}\,P.V.\int_\Omega(\widehat{x-y})\cdot\,\omega(x)\;\,Det\,\big((\widehat{x-y}),\omega(x),\omega(y)\big)\;\tfrac{dy}{|x-\,y|^3}$$

$$-\tfrac{1}{4\,\pi}\,\int_\Omega\,Det\,((\overline{\omega}(x),\,\overline{\omega}(x),\,\omega(y)\,)\;\tfrac{dy}{|x-\,\overline{y}|^3} \tag{2.28}$$

$$-\tfrac{3}{4\,\pi}\,\int_\Omega\Big(\,(\widehat{x-\,\overline{y}})\cdot\,\omega(x)\Big)\;Det\,\Big((\widehat{\overline{x-\,\overline{y}}}),\,\overline{\omega}(x),\,\omega(y)\,\Big)\;\tfrac{dy}{|x-\,\overline{y}|^3}$$

$$+\tfrac{\partial\,\gamma_j(x)}{\partial\,x_i}\,\omega_i(x)\omega_j(x)\,.$$

Clearly the second integral on the right hand side vanishes. We start by proving that

$$I_3(x) =\,0\,,\quad\;\forall x\in\Omega\,. \tag{2.29}$$

By taking into account that $\overline{y} =\,y$ on Γ it follows from (2.26) that

$$\gamma_j(x) = \frac{1}{2\,\pi}\int_\Gamma\epsilon_{jkl}\,\omega_l(y)\,n_k\,\frac{dy}{|x-\,y|}\,. \tag{2.30}$$

Consequently,

$$\frac{\partial\,\gamma_j(x)}{\partial\,x_i} = \frac{1}{2\,\pi}\,P.V.\int_\Gamma\epsilon_{kjl}\,(x_i-\,y_i)\,\omega_l(y)\,n_k\,\frac{dy}{|x-\,y|^3}\,. \tag{2.31}$$

Hence

$$\begin{aligned} I_3(x) =\; &\tfrac{\partial\,\gamma_j(x)}{\partial\,x_i}\,\omega_i(x)\omega_j(x) = \\ &\tfrac{1}{2\,\pi}\,P.V.\int_\Gamma(\widehat{x-\,y})\cdot\,\omega(x)\;\,Det\,(n(y),\omega(x),\omega(y))\;\tfrac{dy}{|x-\,y|^2}\,. \end{aligned} \tag{2.32}$$

Since $n(y)$ and $\omega(y)$ are parallel, (2.29) follows. Hence we may write

$$\begin{aligned} &((\omega\cdot\nabla)\,u\cdot\,\omega)\,(x) = \\ &-\tfrac{3}{4\,\pi}\,P.V.\int_\Omega(\widehat{x-\,y})\cdot\,\omega(x)\;\,Det\,\big((\widehat{x-\,y}),\omega(x),\omega(y)\big)\;\tfrac{dy}{|x-\,y|^3} \\ &-\tfrac{3}{4\,\pi}\,\int_\Omega\Big(\,(\widehat{x-\,\overline{y}})\cdot\,\omega(x)\Big)\;Det\,\Big((\widehat{\overline{x-\,\overline{y}}}),\,\overline{\omega}(x),\,\omega(y)\,\Big)\;\tfrac{dy}{|x-\,\overline{y}|^3} = \\ &I_1+\,I_2\,. \end{aligned} \tag{2.33}$$

Lemma 2.2. *For each $t\in(0,\,T)$ the following estimate holds.*

$$|I_1(t)| \leq\;\frac{\nu}{4}\,|\nabla\,\omega|_2^2+\,c\,\Big(K+\,\nu^{-\frac{3}{5}}\,K^{\frac{4}{5}}|\omega|_2^{\frac{4}{5}}+\,\nu^{-1}\,|\omega|_r^2\Big)\;|\omega|_2^2\,, \tag{2.34}$$

Following [14], we split $\omega(x)$ as

$$\omega(x) =\,\omega^{(1)}(x) +\,\omega^{(2)}(x)$$

where $\omega^{(2)}(x) = 0$ if $|\omega(x)| \leq K$ and $\omega^{(2)}(x) = \omega(x)$ if $|\omega(x)| > K$. Next we replace $\omega(x)$ by $\omega^{(1)}(x) + \omega^{(2)}(x)$ in the expression of I_1. In this way we obtain eight distinct terms, say $\mathcal{K}_{i,j,k}$, indexed in an obvious way, by (i,j,k), $i,j,k = 1,2$. More precisely, $\mathcal{K}_{i,j,k}$ is defined by replacing in the expression of I_1 the first symbol ω by $\omega^{(i)}$, the second by $\omega^{(j)}$ and the third by $\omega^{(k)}$. These eight terms are estimated by following the section 4 in reference [10], with straightforward adaptations. For $(i,j,k) \neq (2,2,2)$ one easily verifies that

$$|\mathcal{K}_{i,j,k}(x)| \leq c\,|\omega^{(i)}(x)|\,|\omega^{(j)}(x)|\,(L\omega^{(k)})(x)\,, \tag{2.35}$$

where

$$(L\omega^{(k)})(x) = \int_\Omega |\omega^{(k)}(y)|\,\frac{dy}{|x-y|^3}\,. \tag{2.36}$$

By the Calderon-Zygmund inequality

$$|L\omega^{(k)}|_q \leq c\,|\omega^{(k)}|_q\,. \tag{2.37}$$

From this last inequality, applied with $q = 2$, it follows that

$$\int_\Omega \mathcal{K}_{i,j,2}(x)\,dx \leq c\,K\,|\omega|_2^2 \tag{2.38}$$

when $(i,j) \neq (2,2)$.

On the other hand, by appealing in particular to (2.37) for $q = 4$, it follows that

$$\int_\Omega \mathcal{K}_{2,2,1}(x)\,dx \leq c\,|\omega^{(1)}|_4\,|\omega|_4\,|\omega|_2\,. \tag{2.39}$$

Since $|\omega|_4 \leq |\omega|_2^{\frac{1}{4}}\,|\nabla\,\omega|_2^{\frac{3}{4}}$ (see [21] Section 1.1, Lemma 2) it readily follows, by appealing to Young's inequality, that the right hand side of (2.39) is bounded by

$$\frac{\nu}{4}\,|\nabla\,\omega|_2^2 + c\,\nu^{-\frac{3}{5}}\,|\omega^{(1)}|_4^{\frac{8}{5}}\,|\omega|_2^2\,.$$

Since $|\omega^{(1)}|_4 \leq K^{\frac{1}{2}}\,|\omega|^{\frac{1}{2}}$ it follows that

$$\int_\Omega \mathcal{K}_{2,2,1}(x)\,dx \leq \frac{\nu}{4}\,|\nabla\,\omega|_2^2 + c\,\nu^{-\frac{3}{5}}\,K^{\frac{4}{5}}|\omega|_2^{\frac{4}{5}}\,|\omega|_2^2\,. \tag{2.40}$$

Next we consider the (2,2,2) term. We write it in the form

(2.41)

$$\mathcal{K}_{2,2,2}(x) =$$

$$-\tfrac{3}{4\,\pi}\,P.V.\int_{\Omega_K}(\widehat{x-y})\cdot\xi(x)\;Det\left((\widehat{x-y}),\xi(x),\xi(y)\right)\;|\omega(x)|^2\,|\omega(y)|\,\tfrac{dy}{|x-y|^3}$$

where, for each fixed t,

$$\Omega_K = \Omega_K(t) = \left\{x \in \mathbf{R}^3_+ : |\omega(x,t)| > K\right\}$$

and, moreover, it is understood that

$$\mathcal{K}_{2,2,2}(x) = 0 \quad \text{if} \quad x \notin \Omega_K\,. \tag{2.42}$$

It readily follows for each $t \in [0, T)$ that

$$|\mathcal{K}_{2,2,2}(x)| \leq \frac{3}{4\pi} P.V. \int_{\Omega} |\sin\theta(x,y)|\, |\omega(x)|^2\, |\omega(y)|\, \frac{dy}{|x-y|^3}\,, \tag{2.43}$$

where (2.42) is always understood. From (2.43) and (1.9) it follows that

$$\int_{\Omega} |\mathcal{K}_{2,2,2}(x)|\, dx \leq \int_{\Omega} c\, |\omega(x)|^2\, I(x)\, dx \tag{2.44}$$

where $I(x)$ is the Riesz potential

$$I(x) = \int_{\Omega} |\omega(y)|\, \frac{dy}{|x-y|^{3-\beta}}\,. \tag{2.45}$$

By a well known Hardy-Littlewood-Sobolev inequality (see [31], Chapter V), if $\beta \in (0,3)$ and $\omega \in L^r(\Omega)$, for some $r \in (1, \frac{3}{\beta})$, then

$$|I(x)|_q \leq c\, |\omega|_r\,, \tag{2.46}$$

where

$$\frac{1}{q} = \frac{1}{r} - \frac{\beta}{3}\,.$$

By this inequality with β and r given by (1.11) it follows that $|I(x)|_3 \leq c\,|\omega|_r$. From equation (2.44) by appealing to Hőlder's inequality (with exponents $3, 2$ and 6) and by a Sobolev's embedding theorem one shows that

$$\int_{\Omega} |\mathcal{K}_{2,2,2}(x)|\, dx \leq \frac{\nu}{4}\, |\nabla\,\omega|^2 + \frac{c}{\nu}\, |\omega|_r^2\, |\omega|_2^2\,. \tag{2.47}$$

By (2.38), (2.40) and (2.47), (2.34) follows.

Lemma 2.3. *For each $t \in (0, T)$ the integral $I_2(t)$ satisfies the estimate* (2.34).

We decompose the I_2 in eight terms, just as done in the previous lemma for I_1. The first seven terms satisfy the same estimates as I_1 do (see (2.38) and (2.40)). It remains to prove that the $(2,2,2)$ term

$$\mathcal{T}_{2,2,2}(x) = -\frac{3}{4\pi} \int_{\Omega_K} \Big((\widehat{x-\overline{y}}) \cdot \omega(x) \Big)\; Det\, \Big((\widehat{x-\overline{y}}),\, \overline{\omega}(x),\, \omega(y) \Big)\, \frac{dy}{|x-\overline{y}|^3} \tag{2.48}$$

satisfies (2.47).

From the boundary condition (1.8) it readily follows that

$$\omega(z) = (0, 0, \omega_3(z))\,, \quad \overline{\omega}(z) = (0, 0, -\omega_3(z)), \quad \forall z \in \Gamma\,. \tag{2.49}$$

Since the solution u is assumed to be regular for $t \in (0, T)$, in this range the assumption (1.9) holds *up to the boundary.*

Define P as the orthogonal projection of $\overline{\Omega}$ onto Γ. From (2.49) one gets

$$\xi(P\,x) = +e_3\,, \quad \text{or} \quad \xi(P\,x) = -e_3\,, \quad \forall x \in \overline{\Omega}\,. \tag{2.50}$$

It readily follows from (2.50) and (1.9) that

$$\sin\angle(\xi(x), \pm\, e_3) \leq c|x_3|^{\beta}\,, \quad \forall x \in \overline{\Omega}\,, \tag{2.51}$$

since $|x - Px| = |x_3|$. The presence of the symbol $\pm$ in an equation means that the equation holds with both signs. The symbol $\angle(a, b)$ denotes the angle between two vectors a and b. Since $\overline{\xi} = (\xi_1, \xi_2, -\xi_3)$, one also has

$$\sin \angle(\overline{\xi}(x), \pm e_3) \leq c|x_3|^\beta, \quad \forall x \in \overline{\Omega}. \tag{2.52}$$

Next we consider the three unit vectors $\overline{\xi}(x)$, $\xi(y)$, and e_3. By identifying the angle $\angle(a, b)$ of two unit vectors a and b with the length of a geodetic on a spherical surface of radius equal to one, one shows that

$$\angle(a, b) \leq \angle(a, c) + \angle(c, b).$$

Consequently, by appealing to (2.52) and to (2.51), with the second equation written with x replaced by y, we prove that

$$\sin \angle(\overline{\xi}(x), \xi(y)) \leq 2\,c|x - \overline{y}|^\beta, \quad \forall x,\, y \in \overline{\Omega}. \tag{2.53}$$

On the other hand (2.48) shows, in particular, that

$$|\mathcal{T}_{2,2,2}(x)| \leq \frac{3}{4\,\pi} \int_{\Omega_K} \sin \angle(\overline{\xi}(x), \xi(y))\, |\omega(x)|^2\, |\omega(y)|\, \frac{dy}{|x - \overline{y}|^3}. \tag{2.54}$$

Due to (2.53), the estimate (2.54) corresponds to the estimate (2.43) in the proof of the previous lemma. Note that $|x - \overline{y}| \leq |x - y|$. Hence we may end the proof as in the previous case.

end of the proof of Theorem 1.1.
From (2.2), (2.33) and lemmas 2.2 and 2.3 it follows that

$$\frac{1}{2}\frac{d}{dt}|\omega|_2^2 + \frac{\nu}{2}\,|\nabla\omega|_2^2 \leq c\, \left(K + \nu^{-\frac{3}{5}}\, K^{\frac{4}{5}} |\omega|_2^{\frac{4}{5}} + \nu^{-1}\, |\omega|_r^2\right)\, |\omega|_2^2. \tag{2.55}$$

Since $|\omega|_2^{\frac{4}{5}}$ and $|\omega|_r^2$ are integrable in $(0, T)$ a well known argument shows that

$$u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2).$$

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