On the viscous Cauchy problem and the existence of shock profiles for a p-sistem with a discontinuous stress function

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Abstract

In this paper, following the ideas introduced in [5], [6] and [7] (cf. also [9] and [3] for related results), we study the existence of weak solutions for the Cauchy problem and the existence of shock profiles for the system in viscoelasticity

$$\begin{cases} v_t - u_x = 0 \\ u_t - \sigma^*(v)_x = \mu u_{xx}, \quad \mu > 0 \end{cases}, \quad x \in \mathbf{R},\ t \geq 0,$$

with $\sigma^*(v) = \sigma(v) + H(v)$, where σ is a smooth stress function as considered in [10] and H is the usual Heaviside function. This kind of models is motivated by some problems in mechanics of solids (cf.[12] and [1]). Finally we solve, in related situations, the Riemann problem for the corresponding hyperbolic system.

1. Introduction and main results.

The existence (and stability) of shock profiles for the system in viscoelasticity

$$\begin{cases} v_t - u_x = 0 \\ u_t - \sigma^*(v)_x = \mu u_{xx}, \quad \mu > 0 \end{cases}, \quad x \in \mathbf{R},\ t \geq 0,$$

has been studied in [10] for nonconvex smooth stress functions σ such that

$$\sigma(0) = 0, \quad \sigma'(0) \geq c > 0,\ \sigma''(v)\, v \geq 0,\ v \in \mathbf{R} \tag{1.1}$$

(ex: $\sigma(v) = v + \frac{v^3}{3}$).

In some problems in mechanics of solids, like the Savart-Masson efect (cf.[12], §3.3.2, and [1], §4.31) it is reasonable to consider models where σ is replaced by

$$\sigma^*(v) = \sigma(v) + H(v), \quad v \in \mathbf{R} \tag{1.2}$$

where $H(v) = 1$ for $v > 0$ and $H(v) = 0$ for $v < 0$ is the usual Heaviside function considered as the multivaluated function

$$\widetilde{H}(v) = H(v) \quad \text{for} \quad v \neq 0, \quad \widetilde{H}(0) = [0,1]. \tag{1.3}$$

A reasonable approximation of H is given by $\int_{-\varepsilon}^{v} \rho_\varepsilon(y)\,dy$, with, for each $\varepsilon > 0$, $\rho_\varepsilon(y) = \frac{1}{\varepsilon}\rho(\frac{y}{\varepsilon})$, where $\rho \geq 0$ is a $\mathcal{D}(\mathbf{R}) = C_c^\infty(\mathbf{R})$ function with support $[-1,1]$ such that $\rho(-y) = \rho(y)$ and $\int_{\mathbf{R}} \rho_\varepsilon(y)dy = 1$ (cf.[5]). The functions ρ_ε are the usual Friedrichs mollifiers and we have $\rho_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \delta$ (Dirac distribution) in $\mathcal{D}'(\mathbf{R})$. We set

$$\sigma_\varepsilon(v) = \sigma(v) + \int_{-\varepsilon}^{v} \rho_\varepsilon(y)\,dy \tag{1.4}$$

and we consider the approximate system

$$\begin{cases} v_{\varepsilon t} - u_{\varepsilon x} = 0 \\ u_{\varepsilon t} - \sigma_\varepsilon(v_\varepsilon)_x = \mu u_{\varepsilon xx} \end{cases}, \quad x \in \mathbf{R},\ t \geq 0 \tag{1.5}$$

The Cauchy problem for systems of the type (1.5) has been studied in [8], [14] and [2] with the hypothesis

$$\frac{\sigma(v)}{\Sigma(v)} \underset{|v|\to\infty}{\longrightarrow} 0, \quad \text{where} \quad \Sigma(v) = \int_0^v \sigma(y)\,dy \tag{1.6}$$

and since

$$\Sigma_\varepsilon(v) = \int_0^v \sigma_\varepsilon(y)\,dy \geq \Sigma(v), \quad v \in \mathbf{R},$$

we get the following result (cf.[14] and [2]) :

Proposition 1 *Assume that the initial data* $(v_0, u_0) \in (H^2(\mathbf{R}))^2$ *and* $v_0 \in L^1(\mathbf{R})$. *Then the Cauchy problem for the system* (1.5) *has a unique global solution*

$$(v_\varepsilon, u_\varepsilon) \in (C^1([0,+\infty[;H^2) \cap C([0,+\infty[;H^2)) \times (C^1([0,+\infty[;L^2) \cap C([0,+\infty[;H^2))$$

such that

$$\int_{\mathbf{R}} (v_\varepsilon^2 + u_\varepsilon^2)(x,t)\,dx \leq c_1, \quad t \geq 0, \tag{1.7}$$

$$\mu \int_0^t \int_{\mathbf{R}} (v_{\varepsilon x}^2 + u_{\varepsilon x}^2)(x,\tau)\,dxd\tau \leq c_2, \quad t \geq 0, \quad 0 < \mu \leq 1, \tag{1.8}$$

$$\mu^2 \int_{\mathbf{R}} v_{\varepsilon x}^2(x,t)\,dx \leq c_3, \quad t \geq 0 \tag{1.9}$$

with c_i, $i = 1,2,3$, *not depending on* ε *(neither on* μ*).*

Remark. From (1.7) and (1.9) we get, for each μ, a uniform (in t and in ε) estimate for $\|v_\varepsilon(\cdot,t)\|_{L^\infty(\mathbf{R})}$.

Now, by the usual techniques in functional analysis, the well known compact imbeddings of Sobolev spaces in bounded domains of $\mathbf{R}^2$ and by lemma 1 in [7], if we let $\varepsilon \to 0$, it is easy to obtain, by diagonalization, a sub-sequence of $(v_\varepsilon, u_\varepsilon)_{\varepsilon>0}$ given by proposition 1, still denoted by $(v_\varepsilon, u_\varepsilon)_{\varepsilon>0}$, such that

$$v_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} v \text{ in } L^\infty(\mathbf{R}_+; L^2) \text{ weak}\star \text{ and a.e. in } \mathbf{R}\times\mathbf{R}_+,$$

$$(v_{\varepsilon x}, v_{\varepsilon t}) \underset{\varepsilon\to 0}{\rightharpoonup} (v_x, v_t) \text{ in } (L^2(\mathbf{R}_+; L^2))^2 \text{ weak}, \quad u_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u \text{ in } L^\infty(\mathbf{R}_+; L^2) \text{ weak}\star,$$

$$u_{\varepsilon x} \underset{\varepsilon\to 0}{\rightharpoonup} u_x \text{ in } L^2(\mathbf{R}_+; L^2) \text{ weak},$$

$$\int_{-\varepsilon}^{v_\varepsilon} \rho_\varepsilon(y)\,dy \underset{\varepsilon\to 0}{\rightharpoonup} \theta \in \widetilde{H}(v) \text{ a.e., and in } L^\infty(\mathbf{R}\times\mathbf{R}_+) \text{ weak}\star$$

and

Theorem 1. *We have* $v \in L^\infty(\mathbf{R}_+; H^1) \cap C([0,+\infty[; L^2)$, $v_x, v_t \in L^2(\mathbf{R}_+; L^2)$, $u \in L^\infty(\mathbf{R}_+; L^2)$, $u_x \in L^2(\mathbf{R}_+; L^2)$, $\theta \in L^\infty(\mathbf{R}\times\mathbf{R}_+)$, $\theta \in \widetilde{H}(v)$ *a.e. and*

$$\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\partial u}{\partial x} = 0 \quad in \quad \mathbf{R}\times\mathbf{R}_+ \\ v(x,0) = v_0(x) \quad a.e.\ in \quad \mathbf{R} \\ \displaystyle\int_0^{+\infty}\int_{\mathbf{R}} u\frac{\partial \psi}{\partial t}\,dxdt - \int_0^{+\infty}\int_{\mathbf{R}} (\sigma(v)+\theta)\frac{\partial\psi}{\partial x}\,dxdt + \int_{\mathbf{R}} u_0(x)\psi(x,0)\,dx = \\ \qquad = \mu\displaystyle\int_0^{+\infty}\int_{\mathbf{R}} \frac{\partial u}{\partial x}\frac{\partial \psi}{\partial x}\,dxdt, \quad \forall\psi\in C_c^1(\mathbf{R}\times[0,+\infty[). \end{cases}$$

Hence, we say that (v,u) is a weak solution of the Cauchy problem for the system

$$\begin{cases} v_t - u_x = 0 \\ u_t - (\sigma(v) + H(v))_x = \mu u_{xx} \end{cases} \tag{1.10}$$

with initial data $(u_0, v_0) \in (H^2(\mathbf{R}))^2$, $v_0 \in L^1(\mathbf{R})$.

Now, in the spirit of [10] (cf. [4] and [13] for related results), we look for the existence of shock profiles for system (1.10) that is special solutions of the form

$$(v,u)(x,t) = (V,U)\left(\frac{x-st}{\mu}\right), \quad (V,U)\in (C(\mathbf{R})\cap L^\infty(\mathbf{R}))^2 \tag{1.11}$$

for given $s \in \mathbf{R}$ such that for certain pairs (v_-, u_-), $(v_+, u_+) \in \mathbf{R}^2$ we have

$$\lim_{\xi \to -\infty} (V(\xi, U(\xi)) = (v_-, u_-), \qquad \lim_{\xi \to +\infty} (V(\xi, U(\xi)) = (v_+, u_+). \tag{1.12}$$

We will choose s, v_-, v_+ such that

$$v_+ < 0 < v_-, \qquad -\sqrt{\frac{\sigma(v_+)}{v_+}} < s < -\sqrt{\sigma'(v_-)} \tag{1.13}$$

(notice that $\dfrac{\sigma(v_+)}{v_+} \leq \sigma'(v_+)$, so (1.13) implies the Lax shock conditions, cf.[11]), and u_-, u_+ such that (Rankine-Hugoniot conditions)

$$\begin{cases} -s(v_+ - v_-) = u_+ - u_- \\ \\ -s(u_+ - u_-) = \sigma(v_+) - \sigma(v_-) - 1 \end{cases} \tag{1.14}$$

(notice that $H(v_+) = 0$, $H(v_-) = 1$). Similarly, it is possible to consider the case $v_- < 0 < v_+$.

Definition 1. *We say that $(v, u) \in (L^\infty(\mathbf{R} \times \mathbf{R}_+))^2$ is a weak solution of system* (1.10) *if there exists $\theta \in L^\infty(\mathbf{R} \times \mathbf{R}_+)$ such that $\theta \in \widetilde{H}(v)$ a.e. and*

$$\begin{cases} v_t - u_x = 0 \\ \\ u_t - (\sigma(v) + \theta)_x = \mu u_{xx} \end{cases}, \quad in \quad \mathcal{D}'(\mathbf{R} \times \mathbf{R}_+) \tag{1.15}$$

For special solutions of the form (1.11), (1.15) can be written as follows, with $\Theta \in L^\infty(\mathbf{R})$, $\Theta \in \widetilde{H}(v)$ a.e.,

$$\begin{cases} sV' + U' = 0 \\ \\ -sU' - (\sigma(V) + \Theta)' = U'' \end{cases}, \quad in \quad \mathcal{D}'(\mathbf{R}) \tag{1.16}$$

(where $V' = \frac{d}{d\xi} V$).

We will prove the following theorem :

Theorem 2. *Under the hypothesis* (1.13), (1.14), *the system* (1.10) *has a special weak solution (v, u) of the form* (1.11) *verifying* (1.12) *(shock profile).*

Now, for each $\mu > 0$, let $(v_\mu, u_\mu)(x, t) = (V, U)\left(\dfrac{x - st}{\mu}\right)$, be a shock profile in the framework of theorem 2 and let $\mu \to 0^+$. For $x \neq st$, $(v_\mu, u_\mu) \underset{\mu \to 0}{\longrightarrow} (v, u)$ defined by

$$v(x,t) = \begin{cases} v_+ & \text{for} \quad x > st \\ \\ v_- & \text{for} \quad x < st \end{cases}, \qquad u(x,t) = \begin{cases} u_+ & \text{for} \quad x > st \\ \\ u_- & \text{for} \quad x < st \end{cases} \tag{1.17}$$

and if $\theta_\mu \in \widetilde{H}(v_\mu)$ is an in definition (1.15) we derive $\theta_\mu \underset{\mu\to 0}{\longrightarrow} \theta$ (for $x \neq st$) with

$$\theta(x,t) = 0 \ \text{ for } \ x > st, \quad \theta(x,t) = 1 \ \text{ for } \ x < st \tag{1.18}$$

that is, $\theta(x,t) = 1 - H(x-st)$, $\theta \in \widetilde{H}(v)$ a.e.. Furthermore, it is easy to deduce (from (1.15) applied to (v_μ, u_μ) and letting $\mu \to 0$)

$$\begin{aligned} &\int_0^{+\infty}\int_{\mathbf{R}} v\varphi_t\,dxdt - \int_0^{+\infty}\int_{\mathbf{R}} u\varphi_x\,dxdt + \int_{\mathbf{R}} v(x,0)\,\varphi(x,0)\,dx + \\ &+ \int_0^{+\infty}\int_{\mathbf{R}} u\psi_t\,dxdt - \int_0^{+\infty}\int_{\mathbf{R}} (\sigma(v)+\theta)\psi_x\,dxdt + \int_{\mathbf{R}} u(x,0)\,\psi(x,0)\,dx = 0 \end{aligned} \tag{1.19}$$

$\forall \varphi, \psi \in C_0^1(\mathbf{R} \times [0, +\infty[)$.

Hence, under the hypothesis of theorem 2, we have solved the corresponding Riemann problem for the discontinuous $p-$system

$$\begin{cases} v_t - u_x = 0 \\ u_t - (\sigma(v) + H(v))_x = 0 \end{cases} \tag{1.20}$$

in the sense introduced in [5]:

Theorem 3. *Assume* (1.13), (1.14). *Then there exists a shock solution* (v,u), *defined by* (1.17), *of the Riemann problem for* (1.20).

2. Proof of theorem 2.

For fixed $\mu > 0$, $\varepsilon > 0$, we look for a solution (we drop the ε for simplicity) (v,u) of the approximate system (1.5) in the form (1.11), $(V,U) \in C^1(\mathbf{R}) \times C^2(\mathbf{R})$ verifying (1.12) under the hypothesis (1.13) and (1.14) by choosing ε small enough such that $\rho_\varepsilon(v_+) = \rho_\varepsilon(v_-) = 0$. System (1.5) reduces to (by setting $\xi = x - st$)

$$\begin{cases} sV' + U' = 0 \\ -sU' - (\sigma'(V) + \rho_\varepsilon(V))\,V' = U'' \end{cases} \tag{2.1}$$

$$s < 0, \quad s^2 = \frac{\sigma(v_+) - \sigma(v_-) - 1}{v_+ - v_-}.$$

As in [10], proposition 3.1, we derive, with σ_ε defined by (1.4),

$$\begin{cases} sV - U = a_1 = -sv_\pm - u_\pm \\ -sU - (\sigma_\varepsilon(V) = U' + a_2, \quad a_2 = -su_\pm - \sigma(v_\pm) - H(v_\pm) \end{cases} \tag{2.2}$$

and so

$$sV' = -s^2V + \sigma_\varepsilon(V) - a, \tag{2.3}$$

with $a = sa_1 - a_2 = -s^2v_\pm + \sigma(v_\pm) + H(v_\pm)$. Hence the function

$$h_\varepsilon(y) = -s^2y + \sigma_\varepsilon(y) - a$$

verify $h_\varepsilon(v_\pm) = 0$. If we put $h(y) = -s^2y + \sigma(y) + H(y) - a$ we deduce

$$h(v_\pm) = 0, \quad h(0^-) = -a, \quad h(0^+) = 1 - a \quad \text{and} \quad a < 0$$

since $s^2 < \dfrac{\sigma(v_+)}{v_+}$. It is easy to see that $h(y) > 0$, $h_\varepsilon(y) > 0$, $y \in]v_+, v_-[$ (cf. fig.1 below).

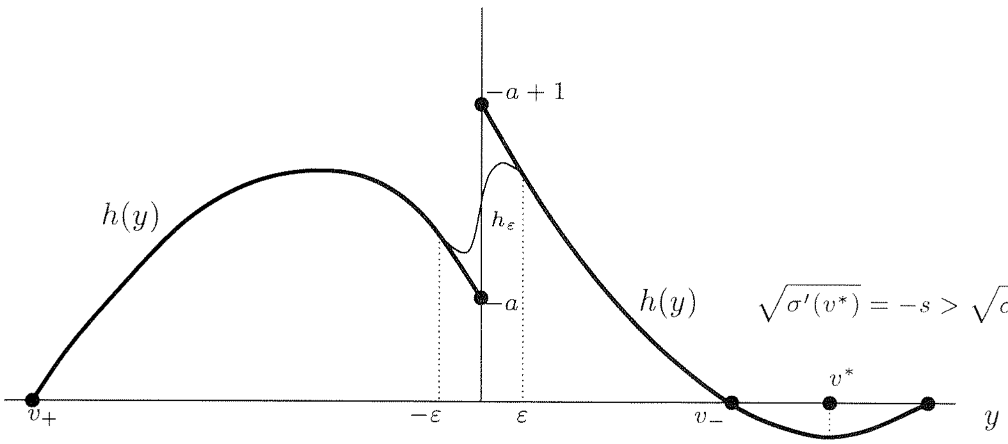

Fig. 1

Hence, following [10], proof of proposition 3.1, a solution V_ε of the approximate problem (2.3) is given by

$$s\int_0^{V_\varepsilon(\xi)} \frac{dy}{h_\varepsilon(y)} = \xi + c_\varepsilon$$

and we can choose $c_\varepsilon = 0$, that is $V_\varepsilon(0) = 0$. By (2.2) we obtain the solution $(V_\varepsilon, U_\varepsilon)$. Now, for each $\varepsilon > 0$ (ε small enough to have $\rho_\varepsilon(v_+) = \rho_\varepsilon(v_-) = 0$) let

$$(v_\varepsilon, u_\varepsilon)(x,t) = (V_\varepsilon, U_\varepsilon)\left(\frac{x - st}{\mu}\right)$$

be the corresponding shock profile. We know that $V_\varepsilon(\xi) \in]v_-, v_+[$, $U_\varepsilon(\xi) \in]u_-, u_+[$. From (2.3), (2.2) and (2.1) we derive that

$$\{(V_\varepsilon, U_\varepsilon)\}_{0<\varepsilon\leq\varepsilon_0} \quad \text{is bounded in} \quad (W^{1,\infty}(\mathbf{R}))^2.$$

Hence, there is a sub-sequence, still denoted $\{(V_\varepsilon, U_\varepsilon)\}_\varepsilon$ and a pair $(V, U) \in (C(\mathbf{R}))^2$, such that $V(0) = 0$, $V(\xi) \in [v_-, v_+]$, $U(\xi) \in [u_-, u_+]$ and

$$V_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} V, \quad U_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} U,$$

uniformly in compact subsets of $\mathbf{R}$. In particular, by lemma 1 in [5], we can assume that

$$\int_{-\varepsilon}^{V_\varepsilon} \rho_\varepsilon(y)\, dy \underset{\varepsilon\to 0}{\rightharpoonup} \Theta \quad \text{in} \quad L^\infty \text{ weak}*\,, \quad \Theta \in \widetilde{H}(V) \text{ a.e..}$$

We derive, by (2.1) written for $(V_\varepsilon, U_\varepsilon)$, by passing to the limit when $\varepsilon \to 0$,

$$\begin{cases} sV' + U' = 0 \\ -sU' - \dfrac{d}{d\xi}(\sigma(V) + \Theta) = U'' \end{cases} \quad \text{in } \mathcal{D}'(\mathbf{R}) \tag{2.4}$$

It is easy to see that, if

$$v_+ < V(\xi) < v_-, \quad \xi \in \mathbf{R}, \tag{2.5}$$

then

$$s\int_0^{V_\varepsilon(\xi)} \frac{dy}{h_\varepsilon(y)} \underset{\varepsilon\to 0}{\longrightarrow} s\int_0^{V(\xi)} \frac{dy}{h(y)}$$

and so

$$s\int_0^{V(\xi)} \frac{dy}{h(y)} = \xi, \quad \xi \in \mathbf{R}.$$

From the properties of h (cf. [10] for related arguments) we derive

$$V(\xi) \underset{\xi\to+\infty}{\longrightarrow} v_+, \quad V(\xi) \underset{\xi\to-\infty}{\longrightarrow} v_-,$$

and so (by passing also to the limit in (2.2), written for $(V_\varepsilon, U_\varepsilon)$, when $\varepsilon \to 0$) we deduce

$$U(\xi) \underset{\xi\to+\infty}{\longrightarrow} u_+, \quad U(\xi) \underset{\xi\to-\infty}{\longrightarrow} u_-.$$

Finnaly, it is easy to conclude that (2.5) holds. This achieves the proof of theorem 2. ∎

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