ON TIME-PERIODIC SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN UNBOUNDED CYLINDRICAL DOMAINS. LERAY'S PROBLEM FOR PERIODIC FLOWS.

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Abstract

Poiseuille flows in infinite cylindrical pipes, in spite of it enormous simplicity, have a main role in many theoretical and applied problems. As is well known, the Poiseuille flow is a stationary solution to the Stokes and the Navier-Stokes equations with a given constant flux. Time-periodic flows in channels and pipes have a comparable importance. However, the problem of the existence of time-periodic flows in correspondence to any given, time-periodic, total flux, is still an open problem. A solution is known only in some very particular cases as, for instance, the Womersley flows. Our aim is to solve this problem in the general case. This existence result open the way to further investigations, in particular by following in the footsteps of the stationary case.

As an application, we present the first steps to the study of Leray's problem for the Stokes and Navier-Stokes equations. We leave to the interested reader, or to forthcoming papers, the adaptation to time-periodic flows of the known techniques already developed for stationary flows.

1 Introduction

We start by giving the motivation that led us to consider the problem below. Let Ω be a bounded, connected open set in $\mathbb{R}^n$, $n \geq 1$, and consider a cylindrical, $(n+1)-$dimensional pipe $\Lambda_+ = \Omega \times \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the positive real line. We denote by Γ the boundary of Ω. We set $x = (x_1, ..., x_n)$ and denote by z the longitudinal coordinate along the axis of the pipe, say $z = x_{n+1}$. We denote by χ the component of the velocity v in the axial direction z. Note that the physical dimension is $N = n + 1$. By assumption, the fluid adheres to the lateral boundary of the cylinder.

Assume that a viscous incompressible fluid is pumped into the pipe Λ_+ with a given inflow velocity $v_0(x,t) = v(x,z,t)_{|z=0}$. The point wise values of the inflow velocity are unknown, and not necessarily time periodic, but the total flux $g(t)$ is a time periodic function, i.e., $\int_\Omega \chi_0(x,t)\,dt = g(t)$, where $\chi_0 = \chi_{|z=0}$. Note that the inflow velocity can be point wisely quite "chaotic" but not the total amount of pumped fluid by unit of time. Note that this is a very natural situation in many physical problems (the blood pumped by the heart, for instance). Clearly, the incompressibility of the fluid implies that

$$\int_\Omega \chi(x,z,t)\,dx = g(t)\,, \qquad (1.1)$$

$z \geq 0$, for each cross section $\Omega(z)$ and at any time $t \geq 0$. We call the flux $g(t)$, in the cross sections of the pipe, the *total flux.* One might guess that after a long time, in a very long pipe, the outflow velocity "forgets" the particular point wise distribution of the inflow velocity v_0, and merely "remembers" the total flux $g(t)$. As we will see, this leads to a non standard variational problem. Contrarily to the stationary case, the *main open problem* is now whether there exists, in an infinite pipe $\Lambda = \Omega \times \mathbb{R}$, a periodic flow with a given time periodic flux $g(t)$. If we assume that a unique limit solution exists, in correspondence to a given g, than the solution must be independent of z.

In spite of the recognized, theoretical and applied, significance of this very basic problem, a positive answer is known only in a very few cases: for instance, the classical Poiseuille steady flow, when the flux is constant; and the Womersley flow, which corresponds to a quite particular but important class of periodic sinusoidal fluxes in circular pipes, see [25]. In both cases the velocity is parallel to the pipe axis.

Let us described other motivations for this research. As in the Womersley paper, we also have in mind flows in large arteries. Here, the heart beat gives rise to a periodic variation, the pulsatility, hence to a time periodic total flux $g(t)$. However, this flux is far from being of sinusoidal type. Nevertheless, in many blood flow simulations, the Womersley model is used, may be due to the lack of information on more general periodic solutions. Concerning blood flow problems see, for instance, [21].

A central motivation for our study is the famous Leray problem. Here, two cylindrical, semi-infinite pipes, Λ_1 and Λ_2, are connected by a reservoir Λ_0. One considers the problem of the existence of a viscous, incompressible fluid flow, subjected to convergence to Poiseuille flows, in both pipes, as the distance goes to infinity. The constant flux g is assigned. A fundamental contribution to Leray's problem is that given by Amick in reference [1], dedicated to Leray himself, and reference [2], to which we refer the interested reader. Leray's problem seems to have been proposed, see [1], by Leray himself to Ladyzhenskaya, who in reference [12] attempted an existence proof under no restrictions on the viscosity. As referenced in [1], this problem is also mentioned by Finn in the review paper [6]. For the Leray's and related problems we refer, in particular, to [8], Vol.I, Chap. VI, sections 1 and 2, and Vol. II, Chap. XI, sections 1, 2, 3 and 4. Other main references are [3], [7], [10], [11], [13], [14], [15], [17], [18], [19], [23]. For the Leray problem concerning non-Newtonian fluids we refer the reader to [20] and references therein.

Note that, due to the arbitrariness of the connection reservoir Λ_0, the "inflow" velocity $v_0(x,t)$ at the second pipe is essentially arbitrary, except for the given constant total flux. Consequently, an intimately related problem, in semi-infinite pipes, is that of the convergence, as the distance goes to infinity, to a Poiseuille flow when a constant (total) inflow flux is given.

The central position occupied by periodic flows in pipes leads one to consider all the above class of problems by replacing the constant flux by an arbitrary time periodic flux $g(t)$. Now, the basic open problem is to prove the existence of a time periodic flow with a given time periodic flux $g(t)$. In the sequel we prove that to each periodic $g(t)$ it corresponds one and only one periodic flow, parallel to the axis, with a given periodic flux. Once the existence of these basic periodic flows is proved, further developments can be done by adapting to the periodic case the known proofs done for the stationary case (see, for instance, [1] and [8]). For this reason, and also to avoid more technical proofs, we will merely take into account some basic problems, and leave to the reader further developments, in particular more stringent results on the asymptotic behavior of the solutions at infinity distance. Other interesting extensions concern problems with more than two exits to infinity and applications to more general fluids.

The main problem is the following: Consider an infinite pipe $\Lambda = \Omega \times \mathbb{R}$ with boundary $\Sigma = \Gamma \times \mathbb{R}$. Let a T-periodic function $g(t)$ be given. We look for T-periodic solutions $v(x, z, t)$ in $\Lambda \times \mathbb{R}$ of the Navier-Stokes equations which are parallel to the z-axis, independent of z, vanish on the boundary Σ and satisfy the flux constraint (1.1). We give a positive answer to this question in Theorem 2.1 below.

After this first result, we consider the Stokes equations (6.2) and prove the existence and uniqueness of the solution to Leray's problem for an arbitrary given time periodic flow $g(t)$. See Theorem 6.1. Finally we assume that $n \leq 4$ and prove the existence of the solution to Leray's problem for the Navier-Stokes equations if the viscosity ν is sufficiently large. See Theorem 7.1.

Without loss of generality, we assume in the sequel that

$$meas\,\Omega = 1\,, \quad \text{and that} \quad T = 2\,\pi\,. \tag{1.2}$$

In order to avoid misunderstandings between z and t, we denote by $\mathbb{R}_t$ the real line when referred to the time variable.

It is worth noting that if we replace the adherence boundary condition by a Neumann type boundary condition (as, for instance, a slip type boundary conditions; see, for instance, [4]) than the above problem becomes trivial.

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2 The existence theorem in infinite pipes

Let Ω be as above and consider the Navier-Stokes equations in the cylindrical domain Λ under the non slip boundary condition on the lateral boundary, namely

$$\begin{cases} \frac{\partial v}{\partial t} - \nu\,\Delta\, v + (v \cdot \nabla)\, v + \nabla p = 0, \\ \\ \nabla \cdot v = 0 \quad \text{in } \Lambda \times \mathbb{R}_t\,; \\ \\ v = 0 \quad \text{in } \Sigma \times \mathbb{R}_t\,. \end{cases} \tag{2.1}$$

Since we look for solutions parallel to the axis of the pipe, we merely consider the longitudinal component χ of the velocity. Moreover, independence of the velocity on z easily implies that the Navier-Stokes equations reduce to

$$
(2.2) \qquad \begin{cases} \frac{\partial \chi}{\partial t} - \nu \Delta \chi + \frac{\partial p}{\partial z} = 0 \,, \\ \frac{\partial p}{\partial x_k} = 0 \,, \quad \text{for}\, k = 1, ..., n \,, \\ \chi_{|\Sigma} = 0 \,. \end{cases}
$$

Hence we look for solutions χ to the problem (2.2) satisfying (2.8) below and also $\chi(t+T) = \chi(t)$.

From the first equation (2.2) it follows that $\frac{\partial p}{\partial z}$ is independent of z. Hence $p(t,z) = a(t) - \psi(t)\, z$. Since the term $a(t)$ does not affect the velocity field, we may assume that the pressure has the form

$$
(2.3) \qquad p(t,z) = -\psi(t)\, z \,.
$$

Note that the significant quantity is here $\nabla p = -\psi(t)\, \vec{e}_z$, where $\vec{e}_z$ denotes the unit vector in the z direction. The pressure gradient in a finite tube of length $l = z_2 - z_1$ is given by $\delta p = -l\, \psi(t)$.

The full problem becomes

$$
(2.4) \qquad \begin{cases} \frac{\partial \chi}{\partial t} - \nu \Delta \chi = \psi(t) \,, \quad in \quad \Omega \times \mathbb{R}_t \,, \\ \chi = 0 \quad on \quad \Gamma \times \mathbb{R} \,, \\ \chi(t+T) = \chi(t) \,, \quad \forall t \in \mathbb{R}_t \,, \end{cases}
$$

together with the constraint (2.8). The unknowns are χ and ψ. Integration of equation (2.4) in Ω shows that we must have

$$
(2.5) \qquad \psi(t) = g'(t) - \nu \int_\Omega \Delta \chi \, dx \,.
$$

See the Remark 2.2 at the end of this section.

Summarizing: Let $g(t)$, $t \in \mathbb{R}_t$, be a given real 2π-periodic function. A 2π-periodic solution v to the Navier-Stokes equations (2.1) in an infinite cylinder Λ, parallel to the axis of the cylinder, i.e., of the form

$$
(2.6) \qquad v(t,x,z) = (0, ..., 0, \chi(t,x)) \,,
$$

and satisfying the flux condition (1.1) for each $t \in \mathbb{R}_t$ exists if and only if χ is a solution to the problem

$$
(2.7) \qquad \begin{cases} \frac{\partial \chi}{\partial t} - \nu \Delta \chi + \nu \int_\Omega \Delta \chi \, dx = g'(t) \,, \quad in \quad \Omega \times \mathbb{R}_t \,, \\ \chi = 0 \quad on \quad \Gamma \times \mathbb{R}_t \,, \\ \chi(t+T) = \chi(t) \quad \forall t \in \mathbb{R}_t \,, \end{cases}
$$

for which

$$
(2.8) \qquad \int_\Omega \chi(x,t) \, dt = g(t) \,,
$$

for each t.

The following existence and uniqueness result of the above solution v is a corollary of Theorem 3.1 below. For notation, see the next section. The symbol # denotes $2\,\pi$-periodicity.

Theorem 2.1. *Let Ω be an open, bounded and connected, set in $\mathbb{R}^n$ and consider the infinite cylinder $\Lambda = \Omega \times \mathbb{R}$. Let $g \in H^1_\#(\mathbb{R}_t)$ be given. There is a unique solution v of the Navier-Stokes equations (2.1) in Λ which satisfies the adherence boundary condition $v_{|\Gamma} = 0$ for each t, and such that:*

v is $(2\,\pi)$-time periodic.

v has the form (2.6).

The total flux satisfies (2.8).

Moreover χ satisfies the estimates

$$\|\Delta\,\chi\|^2_{L^2_\#(\mathbb{R}_t;H)} \leq c\,\|g\|^2_{L^2_\#(\mathbb{R}_t)} + \frac{c}{\nu^2}\,\|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,, \tag{2.9}$$

$$\|\chi'\|^2_{L^2_\#(\mathbb{R}_t;H)} \leq c\,\nu^2\,\|g\|^2_{L^2_\#(\mathbb{R}_t)} + c\,\|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,, \tag{2.10}$$

and

$$\|\chi\|^2_{C_\#(\mathbb{R}_t;V)} \leq c\,(1+\nu)\,\|g\|^2_{L^2_\#(\mathbb{R}_t)} + c\left(\frac{1}{\nu} + \frac{1}{\nu^2}\right)\|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,. \tag{2.11}$$

Remark 2.1. *If Ω is locally situated on one side of Γ and if Γ is a differentiable manifold of class $C^{1,1}$, or if Ω is convex, then, by well known elliptic regularity results,*

$$\|\Delta\,\chi\|^2_{L^2_\#(\mathbb{R}_t;H)} \simeq \|\chi\|^2_{L^2_\#(\mathbb{R}_t;H^1_0\cap H^2)}\,.$$

Moreover, well know regularity results for the heat equation, yield regularity results for χ and v, depending on the regularity of Γ and $g(t)$. In particular, if Γ and $g(t)$ are infinitely differentiable, so is v in $\overline{\Lambda} \times \mathbb{R}$.

Clearly, partial derivatives of v of any order vanish if they include differentiation with respect to z. Otherwise, derivatives are not integrable (with any exponent) in the whole of the cylinder Λ.

Remark 2.2. *If we assign the pressure gradient $-\psi(t)$ instead of the total flux $g(t)$, see (2.4), existence, uniqueness and estimates for the solution are immediate. However, it is worth noting that an estimate of $g(t)$ simply follows from the knowledge of the volume of fluid pumped into the pipe. On the contrary, $\psi(t)$ is a typical "outflow product", that can not be measured at the inflow, at least in real problems. This simple fact is connected to the difficulty of obtaining an explicit relation between ψ and g alone; see (2.5).*

3 Functional framework. An abstract result

The above problem can be easily seen as a particular case of a more general class of problems. In our opinion a little more "abstract" presentation helps in the understanding of the problem.

In the sequel classical and universally accepted notations will be used without definition. Set

$$H = L^2(\Omega); \quad V = H_0^1(\Omega)\,,$$

$$A = -\Delta$$

with domain $D(A) = \{v \in V : A\,v \in H\}$. Note that

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega)\,,$$

if Ω is of class $C^{1,1}$, or convex.

We denote respectively by $(\;,\;)$ and $\|\;\|$ the scalar product and the norm in H. In the two last sections this same notation will be used to denote the scalar product and the norm in different L^2 spaces, related to $n+1$ dimensional domains.

Note that A is an isomorphism between $D(A)$ and H, where the norm of an element $v \in D(A)$ is given by $\|A\,v\|$.

V and H are real separable Hilbert spaces, with V densely and compactly embedded in H. We identify H with its dual H'. If V' denotes the dual of V we have $V \subset H \simeq H' \subset V'$. Moreover A is defined by means of $(Au, v) = a(u, v)$, where a is a symmetric, continuous, bilinear, V-elliptic form over $V \times V$.

In this more general context, we take $a(u, v)$ as the scalar product in V, and set $((u, v)) = a(u, v)$. Hence, $(Au, v) = ((u, v))$; moreover

$$(Av, v) = \|v\|_V^2\,,$$

for each $v \in V$.

We denote by e the constant function $e(x) = 1$, for each $x \in \Omega$. The function $w \in D(A)$ is defined as the unique solution of the equation

$$A\,w = e\,. \tag{3.1}$$

We set

$$C_1^2 = (Aw, w) = \|w\|_V^2\,,$$

and

$$C_0^2 = \|w\|^2\,.$$

In the corresponding "abstract" setting, e represents a fixed $e \in H$, such that $\|e\| = 1$ and $e \notin V$. The assumption $\|e\| = 1$ is strictly connected to the assumption (1.2). In fact, if we assume that $\|e\| > 1$ our problem has, in general, no solution. On the other hand, if $\|e\| < 1$, the existence and uniqueness of the solution is trivial.

We set $L^2_\#(\mathbb{R}_t) = L^2_\#(\mathbb{R}_t; \mathbb{R})$ and $H^1_\#(\mathbb{R}_t) = H^1_\#(\mathbb{R}_t; \mathbb{R})$.

In the above, more general framework, the problem (2.7) becomes:

Problem. Let e be as above and $g(t)$ be a given real, 2π-time periodic function. We look for solutions χ to the linear problem

$$\begin{cases} \chi' + \nu\,A\,\chi - \nu\,(A\,\chi, e)\,e = g'(t)\,e\,, \\ \chi(t+T) = \chi(t)\,, \end{cases} \tag{3.2}$$

such that

$$(\chi(t), e) = g(t)\,. \tag{3.3}$$

The next two sections are dedicated to proving the following theorem. Recall that the symbol # means $2\,\pi$-periodicity.

Theorem 3.1. *Let $g \in H^1_\#(\mathbb{R}_t)$ and $e \in H$, $\|e\| = 1$ and $e \notin V$, be given. Then there is a unique solution χ to the problem (3.2) such that (3.3) holds. One has $\chi \in L^2_\#(\mathbb{R}_t; D(A)) \cap C_\#(\mathbb{R}_t; V)$, $\chi' \in, L^2_\#(\mathbb{R}_t; H)$ and*

$$\|\chi\|^2_{L^2_\#(\mathbb{R}_t;D(A))} \leq \widetilde{C}_0\, \|g\|^2_{L^2_\#(\mathbb{R}_t)} + \frac{\widetilde{C}}{\nu^2}\, \|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,. \tag{3.4}$$

Moreover,

$$\|\chi'\|^2_{L^2_\#(\mathbb{R}_t;H)} \leq 8\,\widetilde{C}_0\,\nu^2\, \|g\|^2_{L^2_\#(\mathbb{R}_t)} + (2 + 8\,\widetilde{C})\, \|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,, \tag{3.5}$$

where $\widetilde{C}_0 = max \left\{\widetilde{C},\, C_1^{-4}\right\}$. In particular,

$$\|\chi\|^2_{C_\#(\mathbb{R}_t;V)} \leq c\,(1 + \nu)\, \|g\|^2_{L^2_\#(\mathbb{R}_t)} + c\, \left(\frac{1}{\nu} + \frac{1}{\nu^2}\right)\, \|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,. \tag{3.6}$$

Note that the map $\chi \to \nu\, A\,\chi - \nu\,(A\,\chi, e)\,e$ is not defined in V, since $e \notin V$. However, even if e should belong to V, the canonical variational techniques, in the functional framework of V, are not recommended here.

Note that $\|f\|^2 = (f, e)^2$ if and only if $f = c\,e$, for some constant c. Consequently,

$$\|A\,\chi\|^2 = (A\,\chi, e)^2 \qquad \Leftrightarrow \qquad \chi = c\,w\,.$$

Uniqueness of the solution obvious. In fact, let χ be a periodic solution to the homogeneous problem

$$\chi'(t) + \nu\,A\,\chi - \nu\,(A\,\chi, e)\,e = 0\,. \tag{3.7}$$

Scalar multiplication by $A\,\chi$ followed by integration on $(0, 2\,\pi)$ shows that $\|A\,\chi\|^2 = (A\,\chi, e)^2$ a.e. in $(0, 2\,\pi)$, as follows from

$$(\chi', A\,\chi) = \frac{1}{2}\,\frac{d}{dt}\,\|\chi\|^2_V$$

and from the periodicity of $\|\chi\|_V$. Hence $\chi = c(t)\,w$. If, moreover, χ satisfies (3.3) with $g = 0$, then $c(t)$ must vanish identically.

We state the following Lemma in the form needed in the sequel.

Lemma 3.2. *If*

$$v' \in L^2((a,b); H) \quad \text{and} \quad v \in L^2((a,b); D(A))\,, \tag{3.8}$$

then $v \in C([a,b]; V)$, moreover

$$\|v\|^2_{C([a,b];V)} \leq 8\, \left[\frac{2}{b-a}\, \|v\|_{L^2(a,b;H)} + \|v'\|_{L^2(a,b;H)}\right]\, \|v\|_{L^2(a,b;D(A))}\,. \tag{3.9}$$

Proof

The fact that $v \in C([a,b];V)$, together with a suitable estimate, is a very particular case of well known results. See [16], Chapter 1, section 3.1 and [5], Chapter XVIII, Section 1.3. Here, we want just show the estimate (3.9). It is well known, see [16], Chapter 1, section 2.4, Proposition 2.1, that $[D(A),\, H]_{1/2} = V$. Moreover,

$$\frac{1}{2}\,\frac{d}{dt}\,\|v\|_V^2 = \,(v', A\,v)\,. \tag{3.10}$$

If $v(0) = 0$, it readily follows that

$$\|v\|_{C([a,b];V)} \leq 2\,\|v'\|_{L^2(a,b;H)}\,\|A\,v\|_{L^2(a,b;H)}\,.$$

In the general case we apply the above estimate to the functions $\alpha\,v$ and $(1-\alpha)\,v$, where the real function α belongs to $C^\infty([a,\,b])$, vanishes near a and takes values in $[0,\,1]$. Since $v = \alpha\,v + (1-\alpha)\,v$, the thesis follows easily.

4 An auxiliary problem

In this section we solve the following stationary system in H:

$$\begin{cases} k\,v + \nu\,A\,u - \nu\,(A\,u,e)\,e = k\,q\,e\,, \\ \\ -k\,u + \nu\,A\,v - \nu\,(A\,v,e)\,e = -k\,p\,e\,, \end{cases} \tag{4.1}$$

where $k \geq 1$, p and q are given reals. We want to prove the following result:

Theorem 4.1. *Problem* (4.1) *has one and only one solution* $(u,v) \in D(A) \times D(A)$*. Moreover,*

$$\|Au\|^2 + \|Av\|^2 \leq \widetilde{C}\,\left(1 + \left(\frac{k}{\nu}\right)^2\right)\,(p^2 + q^2)\,, \tag{4.2}$$

where $\widetilde{C}$ depends only on C_0 and C_1 (a simple explicit expression is easily obtained).

Proof. The eigenvalues of the compact operator A form an increasing sequence of strictly positive reals, $\lambda_j\,,\, j = 1,2,...,$

$$A\,w_j = \lambda_j\,w_j\,.$$

The eigenfunctions w_j's are a Hilbertian basis in H; moreover we can assume

$$(w_i, w_j) = \delta_{i\,j}\,.$$

Note that the w_j's are an orthogonal basis in V. More precisely, $((w_i, w_j)) = \delta_{i\,j}\,\lambda_i\,\lambda_j$.

We set $V_m =$ span $\{v_1, v_2, ..., v_m\}$ and look for $u_m,\, v_m \in V_m$ such that

$$\begin{cases} (k\,v_m + \nu\,A\,u_m - \nu\,(A\,u_m,e)\,e,\phi) = k\,q\,(e,\phi)\,, \\ \\ (-k\,u + \nu\,A\,v - \nu\,(A\,v,e)\,e,\phi) = -k\,p\,(e,\phi)\,, \end{cases} \tag{4.3}$$

for each $\phi \in V_m$. Since the $\lambda_l\, w_l$, $l = 1, ..., m$, form a basis of V_m, the problem (4.3) is equivalent to the system of $2m$ equations obtained by replacing the ϕ's by the above $\lambda_l\, w_l\,, l = 1, ..., m$. Note that, formally, this corresponds to multiplication of the equations by $A\, w_l$ (and not by w_l, as usual). Clearly, we look for u_m and v_m of the form

$$u_m = \sum_1^m \alpha_j\, w_j\,; \quad v_m = \sum_1^m \beta_j\, w_j\,. \tag{4.4}$$

Straightforward calculation show that (4.3) is equivalent to the $2m$ dimensional system
(4.5)

$$\begin{cases} k\,\lambda_l\,\beta_l + \nu\,\sum_1^m \lambda_j\,\alpha_j\,(w_j, w_l) - \nu\,\sum_1^m \lambda_j\,\alpha_j\,(w_j, e)\,(e, w_l) = k\,q\,(e, w_l)\,, \\ \\ -k\,\lambda_l\,\alpha_l + \nu\,\sum_1^m \lambda_j\,\beta_j\,(w_j, w_l) - \nu\,\sum_1^m \lambda_j\,\beta_j\,(w_j, e)\,(e, w_l) = -k\,p\,(e, w_l)\,, \end{cases}$$

where l runs from 1 to m. It is convenient to interpret (4.3) as a system on the unknown $2m$-dimensional column vector

$$X = (\lambda_1\,\alpha_1, ..., \lambda_m\,\alpha_m, \lambda_1\,\beta_1, ..., \lambda_m\,\beta_m)\,.$$

To show that the $2m \times 2m$ matrix of the above system is positive definite, and to obtain a suitable estimate, we multiply the first m equations by $\lambda_l\,\alpha_l$, the last m equations by $\lambda_l\,\beta_l$, and sum up for $l = 1, ..., m$. This is equivalent to multiplying the above system, on the left, by the transpose of X. One gets

$$\begin{cases} \nu\,\sum_{j,l=1}^m (w_j, w_l)\,\big((\lambda_j\,\alpha_j)\,(\lambda_l\,\alpha_l) + (\lambda_j\,\beta_j)\,(\lambda_l\,\beta_l)\big) \\ \\ -\,\nu\,\sum_{j,l=1}^m (w_j, e)\,(e, w_l)\,\big((\lambda_j\,\alpha_j)\,(\lambda_l\,\alpha_l) + (\lambda_j\,\beta_j)\,(\lambda_l\,\beta_l)\big) = \\ \\ k\,\sum_{l=1}^m \lambda_l\,(e, w_l)\,(q\,\alpha_l - p\,\beta_l)\,. \end{cases} \tag{4.6}$$

The positivity of the $2m\times 2m$ quadratic form on the left hand side of (4.6) follows from that of the $m \times m$ matrix with coefficients $\gamma_{ij} = \delta_{ij} - (w_j, e)\,(e, w_l)$. It is worth noting that the strict positivity of this last matrix follows from the fact that $e \notin V_m$. Denote by $\overline{e}$ the orthogonal projection of e onto V_m. Then with a clear notation,

$$\sum \gamma_{jl}\,\xi_j\,\xi_l = |\xi|^2 - (\xi, \overline{e})\,(\overline{e}, \xi) \geq (1 - |\overline{e}|^2)\,|\xi|^2\,,$$

for each $\xi \in \mathbb{R}^m$. We have proved that the problem (4.3) admits one and only one solution in $V_m \times V_m$. Moreover, equation (4.6) shows that

$$\begin{aligned} &\nu\,\|A\,u_m\|^2 + \nu\,\|A\,v_m\|^2 - \nu\,\left[(A\,u_m, e)^2 + (A\,v_m, e)^2\right] = \\ &k\,q\,(A\,u_m, e) - k\,p\,(A\,v_m, e)\,. \end{aligned} \tag{4.7}$$

Hence

$$\|A\,u_m\|^2 + \|A\,v_m\|^2 \leq \tfrac{k^2}{4\,\nu^2}\,(p^2 + q^2) + 2\,\left[(A\,u_m, e)^2 + (A\,v_m, e)^2\right]\,. \tag{4.8}$$

On the other hand, for each $\phi \in W$,

$$(A\,\phi - (A\,\phi, e)\,e,\, w) = (\phi, e) - C_1^2\,(A\,\phi, e)\,,$$

and also

$$\|A\,\phi - (A\,\phi, e)\,e\|^2 = \|A\,\phi\|^2 - (A\,\phi, e)^2\,.$$

Consequently,

$$C_1^4\,(A\,\phi, e)^2 \le 2(\phi, e)^2 + 2\,C_0^2\,\left[\|A\,\phi\|^2 - (A\,\phi, e)^2\right]\,, \tag{4.9}$$

Hence, from (4.7), and by appealing to (4.9), one proves that

$$\begin{aligned} &C_1^4\,\left[(A\,u_m, e)^2 + (A\,v_m, e)^2\right] \le 2\left[(u_m, e)^2 + (v_m, e)^2\right] + \\ &2\,C_0^2\,\tfrac{k}{\nu}\,\left[q\,(A\,u_m, e) - p\,(A\,v_m, e)\right]\,. \end{aligned} \tag{4.10}$$

Now, we turn back to the system (4.3). By setting $\phi = \overline{e}$ in both equations, straightforward calculations show that

$$\begin{cases} (v_m, e) = q\,|\overline{e}|^2 - \nu\,\frac{1-|\overline{e}|^2}{k}\,(A\,u_m, e)\,, \\ \\ (u_m, e) = p\,|\overline{e}|^2 + \nu\,\frac{1-|\overline{e}|^2}{k}\,(A\,v_m, e)\,. \end{cases} \tag{4.11}$$

By appealing to (4.11), one easily shows from (4.10) that

$$\left[C_1^4 - 4\,\nu^2\,\left(\frac{1-|\overline{e}|^2}{k}\right)^2\right]\,\left[(A\,u_m, e)^2 + (A\,v_m, e)^2\right] \le$$

$$4\,|\overline{e}|^2\,(p^2+q^2) + C_0^2\,\left\{\frac{k^2}{\epsilon\,\nu^2}\,(p^2+q^2) + \epsilon\,\left[(A\,u_m, e)^2 + (A\,v_m, e)^2\right]\right\}\,,$$

for each positive real ϵ. Note that $\|\overline{e}\|$ converges to 1 as m goes to ∞ and that $k \ge 1$. Hence, in the left hand side of the above inequality, we may replace k by 1 and assume that m is sufficiently large that the coefficient under square brackets is larger than $\frac{C_1^4}{4}$. Hence, by setting $\epsilon = \frac{C_1^4}{2\,C_0^2}$, one shows that

$$C_1^4\,\left[(A\,u_m, e)^2 + (A\,v_m, e)^2\right] \le 8\,\left[2 + \left(\frac{C_0}{C_1}\right)^4\,\left(\frac{k}{\nu}\right)^2\right]\,(p^2+q^2)\,. \tag{4.12}$$

From this last estimate, together with (4.8), we easily obtain (4.2) with u and v replaced by u_m and v_m, respectively. From this estimate follows the weak convergence in $D(A) \times D(A)$ of the pair (u_m, v_m) to a solution (u, v) of (4.1). Clearly, (4.2) holds.

5 Proof of Theorem 3.1: Existence of the periodic solution in an infinite cylinder

In the following we look for the solution $\chi \in L^2_{\#}(\mathbb{R}_t; D(A))$ to the problem (3.2), (3.3) into the form

$$\chi(t) = a_0 + \sum_{k=1}^{\infty} a_k\,\cos k\,t + \sum_{k=1}^{\infty} b_k \sin k\,t\,, \tag{5.1}$$

where the unknowns a_k and b_k belong to $D(A)$.

The data $g \in L^2_{\#}(\mathbb{R})$ is written in the form

$$g(t) = p_0 + \sum_{k=1}^{\infty} p_k \cos k\, t + \sum_{k=1}^{\infty} q_k \sin k\, t\,, \tag{5.2}$$

where the p's and q's are constants.

Substitution in equation (3.2) yields

$$A\, a_0 - (A\, a_0, e)\, e = 0\,, \tag{5.3}$$

together with

$$\begin{cases} k\, b_k + \nu\, A\, a_k - \nu\, (A\, a_k, e)\, e = k\, q_k\, e\,, \\ \\ -k\, a_k + \nu\, A\, b_k - \nu\, (A\, b_k, e)\, e = -k\, p_k\, e\,, \end{cases} \tag{5.4}$$

where k runs from 1 to ∞. Equation (5.3) is equivalent to

$$a_0 = \widetilde{c}\, w\,, \tag{5.5}$$

where $\widetilde{c}$ is an arbitrary constant. We anticipate that the value of the constant $\widetilde{c}$ will be uniquely determined by the constraint (3.3).

On the other hand, each of the infinite systems (5.4), $k \in \mathbb{N}$, has the form (4.1). Theorem 4.1 shows that the coefficients a_k and b_k are uniquely determined. Moreover, the estimate (4.2) shows that

$$\|A\, a_k\|^2 + \|A\, b_k\|^2 \leq \widetilde{C}\, (1 + (\frac{k}{\nu})^2)\, (p_k^2 + q_k^2)\,, \tag{5.6}$$

for each $k \in \mathbb{N}$. On the other hand,

$$A\, \chi(t) = \widetilde{c}\, e + \sum_{k=1}^{\infty} (A\, a_k) \cos k\, t + \sum_{k=1}^{\infty} (A\, b_k) \sin k\, t\,. \tag{5.7}$$

It readily follows from (5.7) that

$$\|\chi\|^2_{L^2_{\#}(\mathbb{R}_t; D(A))} = \int_0^{2\,\pi} (A\, \chi(t), A\, \chi(t))_H\, dt = 2\,\pi\, \widetilde{c}^2 + \pi \sum_{k=1}^{\infty} \left(\|A\, a_k\|^2 + \|A\, b_k\|^2 \right)\,.$$

Finally, by appealing to (5.6),

$$\|\chi\|^2_{L^2_{\#}(\mathbb{R}_t; D(A))} \leq 2\,\pi\, \widetilde{c}^2 + \widetilde{C}\, \pi \sum_{k=1}^{\infty} (p_k^2 + q_k^2) + \widetilde{C}\, \pi\, \nu^{-2} \sum_{k=1}^{\infty} k^2\, (p_k^2 + q_k^2)\,. \tag{5.8}$$

Now we impose (3.3) by choosing the value of the arbitrary constant $\widetilde{c}$. By scalar multiplication of both sides of equation (3.2) by e one shows that

$$\frac{d}{dt} [(\chi, e) - g(t)] = 0\,. \tag{5.9}$$

On the other hand, by (5.1) and (5.5) it follows that

$$(\chi(t), e) = \widetilde{c}\, (w, e) + \sum_{k=1}^{\infty} (a_k, e) \cos kt + \sum_{k=1}^{\infty} (b_k, e) \sin kt\,. \tag{5.10}$$

Differentiation with respect to t of this last equation and of equation (5.2), and appeal to equation (5.9), show that we must have $(a_k, e) = p_k$ and $(b_k, e) = q_k$. Hence, from (5.10) and (5.2), it follows

$$(\chi(t), e) = \widetilde{c}\,(w, e) - p_0 + g(t)\,. \tag{5.11}$$

Consequently, to get (3.3), we have to impose that $\widetilde{c} = \frac{p_0}{C_1^2}$. This shows that (3.3) holds if and only if in (5.1) we set

$$a_0 = \frac{p_0}{C_1^2}\, w\,.$$

Finally, from (5.8) we get

$$\|\chi\|^2_{L^2_\#(\mathbb{R}_t;D(A))} \leq\ 2\,\pi\,\frac{p_0^2}{C_1^4} + \widetilde{C}\,\pi\,\sum_{k=1}^{\infty}(p_k^2 + q_k^2) + \frac{\widetilde{C}}{\nu^2}\|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,. \tag{5.12}$$

This proves (3.4). The estimate (3.5) follows from (3.4) together with the first equation (3.2). Finally, the estimate (3.6) follows from (3.4), (3.5) and (3.9). In (3.9) we estimate $\|v\|_{L^2(a,b;H)}$ simply by $c\,\|v\|_{L^2(a,b;D(A))}$.

6 The Leray's problem. Stokes case.

The aim of this and of the next section is to show that the results proved in the previous sections can be applied to study Leray's problem in the periodic case. More sophisticated results, as well as extensions to more general cases, can be done by adapting the well known proofs followed in the stationary case. In particular, more stringent results on the asymptotic behavior as z goes to infinity; extension to more then two exit pipes; and consideration of non Newtonian fluids. We leave these improvements to the interested reader.

Here Θ is an unbounded, connected open subset of $\mathbb{R}^{n+1}$, locally situated on one side of its boundary, consisting of a "reservoir" Λ_0 with two cylindrical exits to infinity, namely Λ_1 and Λ_2. We denote by $\overline{x} = (x_1, ..., x_n, x_{n+1})$ the system of space coordinates in $\mathbb{R}^{n+1}$. The two semi-infinite pipes can be described, possibly in two different systems of coordinates, in the form $\Lambda_i = \Omega_i \times \mathbb{R}_+$, where the sections Ω_i may have different shape and measure. In this framework, we denote by $z \in \mathbb{R}_+$ the axial coordinate in both cylinders and set $x = (x', z)$. Obviously, in this last case $x' = (x_1, ..., x_n)$ does not denote the same $(x_1, ..., x_n)$ that appears in the above definition of $\overline{x}$.

We set

$$\Lambda_i^r = \{(x, z) \in \Lambda_i :\ z < r\}\ ,$$

moreover,

$$\Theta_r = \Lambda_0 \cup \Lambda_1^r \cup \Lambda_2^r\,.$$

Define

$$\mathcal{V} = \{v \in C_0^\infty(\Theta) :\ \nabla\cdot v = 0\}\ ,$$

and denote respectively by $\mathbb{H}$ and $\mathbb{V}$ the closure of $\mathcal{V}$ in $L^2(\Theta)$ and $H^1(\Theta)$. The scalar products in the spaces $\mathbb{H}$ and $\mathbb{V}$ are denoted respectively by

$$(u, v) = \int_\Theta u\cdot v\,d\overline{x} \qquad \textit{and} \qquad ((u, v)) = (\nabla u, \nabla v) = \int_\Theta \nabla u\cdot\nabla v\,d\overline{x}\,.$$

In particular

$$\mathbb{V} = \left\{ v \in H_0^1(\Theta) : \nabla \cdot v = 0 \right\} .$$

Due to the structure of the unbounded set Θ, Poincaré's inequality $\|v\| \leq \widetilde{c}\,\|\nabla v\|$ holds for each $v \in H_0^1(\Theta)$. In particular,

$$\|v\|_{\mathbb{H}} \leq \widetilde{c}\,\|v\|_{\mathbb{V}}, \tag{6.1}$$

for each $v \in \mathbb{V}$. Hence, in $\mathbb{V}$, the Dirichlet norm $\|v\|_{\mathbb{V}} = \sqrt{((v,v))}$ is equivalent to the the canonical $H^1(\Theta)$ norm.

Denote by $\chi_i(x,t)$, for $i = 1$ and $i = 2$, the basic time periodic flows described in the Theorem 3.1 in connection with the sections $\Omega = \Omega_i$ and with a given, arbitrary, periodic flux $g(t)$. Set $\chi_i(x,z,t) = (0, ..., 0, \chi_i(x,t))$. For convenience we denote $\chi_i(x,z,t)$ simply by $\chi_i(x,t)$.

We look for solutions to the following problem:

Problem PL: given a real $(2\,\pi)$-time-periodic function $g(t)$ find a $(2\,\pi)$-time-periodic function $v(t,x,z)$ of the Stokes evolution problem

$$\begin{cases} \frac{\partial v}{\partial t} - \nu\,\Delta\, v + \nabla\, p = 0\,, \\ \nabla \cdot v = 0\,, \quad in \quad \Theta \times \mathbb{R}_t\,; \\ v = 0 \quad on \quad (\partial\,\Theta) \times \mathbb{R}_t\,, \\ v(t + 2\,\pi) = v(t) \quad \forall t \in \mathbb{R}_t\,, \end{cases} \tag{6.2}$$

such that

$$\sup_{t \in \mathbb{R}_t} \|v(t) - \chi_i(t)\|_{H^1(\Lambda_i)} \leq constant\,, \quad i = 1, 2\,. \tag{6.3}$$

Remark 6.1. *We will show that* (6.16) *holds, where* $u(t) = v(t) - \chi_i(t)$, $i = 1,\, 2$, *in* Λ_i.

The constraint (6.3) implies convergence of v to the χ_i's, as the coordinate z go to infinity, uniformly with respect to t. In fact, a straightforward argument (see, for instance, [8]) shows that

$$\lim_{z \to +\infty} \|v(t) - \chi_i(t)\|_{H^{\frac{1}{2}}(\Omega_i)} = 0 \tag{6.4}$$

uniformly with respect to t.

The solution v of problem (PL) has the form $v = v_0 + u$, where v_0 is an auxiliary flow which coincides on the exit pipes Λ_i with the basic periodic flows χ_i and u is a perturbation of v_0 that goes to zero when the distance z along the exit pipes goes to infinity. More precisely, one has the following result:

Theorem 6.1. *Let* $g \in H^1_{\#}(\mathbb{R}_t)$ *be given. There is a unique solution* v *to problem* (6.2), (6.3). *The solution* v *can be written in the form* $v = v_0 + u$, *where* v_0 *is a solution to the problem* (6.6) *and* u *solves* (6.9). *The flows* v_0 *and* u *satisfy, respectively, the estimates* (6.7) *and* (6.16). *Moreover,* $u = v - \chi_i \quad in \quad \Lambda_i\,, \quad i = 1, 2$, *satisfies the asymptotic estimate* (6.4).

Remark 6.2. *Further regularity results are easily proved. In particular, if Ω is* regular *(say, of class $C^{1,1}$, or convex), then*

$$D(\mathcal{A}) = H^2(\Theta) \cap \mathbb{V}(\Theta)\,, \tag{6.5}$$

and if Θ and g are of class C^∞ so is v. For the definition of $\mathcal{A}$ see the end of this section.

It is worth noting that the convergence of the solution v to the limit functions χ_i, as z goes to ∞, is stronger than that implied by (6.3) alone. If the data are regular, exponential decay should occur, as for the case of the Poiseuille flow. See [1] and [8], Sections VI.1 and VI.2.

For convenience, on writing some of the main estimates, we consider the explicit case in which (6.5) holds. However, the results below hold in the general case, by merely replacing $H^2(\Theta)$ by $D(\mathcal{A})$.

The first step of the proof of Theorem 6.1 consists in constructing a time periodic vector field v_0 in Θ such that

$$\begin{cases} v_0 \in L^2_{\#}(\mathbb{R}_t; H^2(\Theta_1))\,, \\ v_0' \in L^2_{\#}(\mathbb{R}_t; L^2(\Theta_1))\,; \\ \nabla \cdot v_0 = 0 \quad in \quad \Theta\,; \\ v_0 = 0 \quad on \quad (\partial\,\Theta) \times \mathbb{R}_t\,; \\ v_0(t+2\pi) = v_0(t)\,; \\ v_0 = \chi_i \quad in \quad \Lambda_i\,, \quad i = 1, 2\,. \end{cases} \tag{6.6}$$

The construction of the vector field v_0 is done by freezing the variable t. This construction is done by following that of the extended Poiseuille vector field q in reference [1]. See also [23] and [8], Chapter VI, section 1. For details the reader is referred to these references. Following the notation in [8], we chose the truncation functions $\zeta_i(z) \in C_0^\infty(\mathbb{R}^{n+1})$, $i = 1, 2$, equal to 1 in $\Lambda_i - \Lambda_i^1$ and vanishing on $\Theta - \Lambda_i$. The map $(\chi_1, \chi_2) \to v_0$ is linear, moreover

$$\|v_0\|_{H^2(\Theta_1)} \leq c\,(\|\chi_1\|_{H^2(\Lambda_1^1)} + \|\chi_2\|_{H^2(\Lambda_2^1)})\,.$$

Similar estimates hold by replacing H^2 by H_0^1 or by L^2. These facts show that

$$\|v_0\|^2_{L^2_{\#}(\mathbb{R}_t; H^2(\Theta_1))} \leq c \sum_{i=1}^{2} \|\chi_i\|^2_{L^2_{\#}(\mathbb{R}_t; H^2(\Lambda_i^1))}\,.$$

The linearity of the map $(\chi_1, \chi_2) \to v_0$ yields

$$\|v_0'\|^2_{L^2_{\#}(\mathbb{R}_t; L^2(\Theta_1))} \leq c \sum_{i=1}^{2} \|\chi_i'\|^2_{L^2_{\#}(\mathbb{R}_t; L^2(\Lambda_i^1))}\,.$$

By appealing to (2.9), (2.10) and (2.11) it follows that

$$\text{(6.7)}\qquad \begin{cases} \|v_0\|^2_{L^2_\#(\mathbb{R}_t;H^2(\Theta_1))} \le c\,(1+\nu^{-2})\,\|g\|^2_{H^1_\#(\mathbb{R}_t)}\,, \\ \\ \|v_0\|^2_{C_\#(\mathbb{R}_t;H^1(\Theta_1))} \le c\,(\nu+\nu^{-2})\,\|g\|^2_{H^1_\#(\mathbb{R}_t)}\,, \\ \\ \|v_0'\|^2_{L^2_\#(\mathbb{R}_t;L^2(\Theta_1))} \le c\,(1+\nu^{2})\,\|g\|^2_{H^1_\#(\mathbb{R}_t)}\,. \end{cases}$$

Next, we look for solutions of problem (6.2) in the form

$$\text{(6.8)}\qquad v = v_0 + u\,.$$

By setting $f(t) = -\left(\frac{\partial v_0}{\partial t} - \nu\,\Delta\,v_0\right)$, the problem (6.2) becomes

$$\text{(6.9)}\qquad \begin{cases} \frac{\partial u}{\partial t} - \nu\,\Delta\,u + \nabla\,p = f(t)\,, \\ \\ \nabla\cdot u = 0\,, \quad in \quad \Theta\times\mathbb{R}_t\,; \\ \\ u = 0 \quad on \quad (\partial\,\Theta)\times\mathbb{R}_t\,, \\ \\ u(t+2\,\pi) = u(t) \quad \forall t\in\mathbb{R}_t\,. \end{cases}$$

Next we exploit the fact that $v_0 = \chi_i$ in Λ_i, where the functions $\chi_i(x,z,t) = \chi_i(x,t)$ satisfy (2.4) with $\psi(t) = \psi_i(t)$ and $\psi_i(t)$ satisfies (2.5) with $\chi(t) = \chi_i(t)$.

By adding $\nabla\,\sum_i(z\,\zeta_i(z)\,\psi_i(t))$ to the pressure term that appears in the left hand side of the first equation (6.9) we show that we can replace $f(t)$ by

$$f(t) = -\left(\frac{\partial\,v_0}{\partial\,t} - \nu\,\Delta\,v_0\right) + \sum_i \frac{\partial}{\partial\,z}\,(z\,\zeta_i(z)\,\psi_i(t))\,.$$

Since $v_0 = \chi_i$ in Λ_i, the first equation (2.4) shows that $f(t)$ vanishes on $\Lambda^1_1\cup\Lambda^1_2$, i.e.,

$$\text{(6.10)}\qquad supp\,f \subset \Theta_1\,.$$

Furthermore, the estimate (6.7) , together with (2.5) and (2.9), shows that

$$\text{(6.11)}\qquad \|f\|_{L^2_\#(\mathbb{R}_t;L^2(\Theta_1))} \le c\,(1+\nu)\,\|g\|_{H^1_\#(\mathbb{R}_t)}\,.$$

Let us set the problem (6.9) in a variational form.

We look for $u\in L^2_\#(\mathbb{R}_t;\,\mathbb{V})$ such that

$$\text{(6.12)}\qquad \frac{d}{d\,t}(u,\,v) + \nu\,((u,\,v)) = (f(t),\,v)\,, \qquad \forall v\in\mathbb{V}$$

in the distributional sense. We denote by $f(t)$ the orthogonal projection of the above $f(t)$ over $\mathbb{H}$. Note that the orthogonal projection of $f(t)$ over $\mathbb{H}^\perp$ is a gradient, which does not affect the solution of the equation (6.12). One has

$$\text{(6.13)}\qquad \|f\|_{L^2_\#(\mathbb{R}_t;\mathbb{H})} \le c\,(1+\nu)\,\|g\|_{H^1_\#(\mathbb{R}_t)}\,.$$

Under the estimate (6.13) it is known that the problem (6.12) admits a unique solution $u \in L^2_{\#}(\mathbb{R}_t; \mathbb{V})$. In fact, due to (6.1), the existence and the uniqueness of a solution $u \in L^2(0, 2\pi; \mathbb{V})$ to the Cauchy problem $u(0) \in \mathbb{H}$ is well known. In particular, $u' \in L^2(0, 2\pi; \mathbb{H})$. Straightforward calculations show that the map $S : u(0) \to u(2\pi)$ is a strict contraction in $\mathbb{H}$. Moreover $S(B) \subset B$, for a sufficiently large ball $B \subset \mathbb{H}$ (see the next section). This proves the existence of a unique fixed point $u(0) = u(2\pi)$. Canonical devices (formally, scalar multiplication in $\mathbb{H}$ of both sides of the first equation (6.9) by u, followed by integration in $(0, 2\pi)$) shows that

$$\nu \|u\|_{L^2_{\#}(\mathbb{R}_t; \mathbb{V})} \leq c \|f\|_{L^2_{\#}(\mathbb{R}_t; \mathbb{H})},$$

where we have used Poincaré's inequality (6.1). Finally, by (6.13),

$$\nu \|u\|_{L^2_{\#}(\mathbb{R}_t; \mathbb{V})} \leq c (1 + \nu) \|g\|_{H^1_{\#}(\mathbb{R}_t)}. \tag{6.14}$$

This same bound holds as well for $\|u'\|_{L^2_{\#}(\mathbb{R}_t; \mathbb{V}')}$, as easily seen from (6.12) and (6.14).

Further regularity: Following a classical technique, we define an unbounded operator $\mathcal{A}$ in $\mathbb{H}$ by setting

$$(\mathcal{A}u, v)_{\mathbb{H}} = ((u, v))_{\mathbb{V}}.$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ consists of the set of elements $u \in \mathbb{V}$ for which the map $v \to ((u, v))_{\mathbb{V}}$ is an element of $\mathbb{H}'$. Hence $\mathcal{A}u \in \mathbb{H}' \cong \mathbb{H}$. The equation (6.12) gives rise to the equation

$$u' + \nu \mathcal{A}u = f(t). \tag{6.15}$$

Scalar multiplication by $\mathcal{A}u$ and integration over $(0, 2\pi)$ show, in a first stage, that $\mathcal{A}u$ satisfies the estimate

$$\begin{aligned} &\|u'\|_{L^2_{\#}(\mathbb{R}_t; \mathbb{H})} + (\nu^{-1} + \nu^{1/2}) \|u\|_{C_{\#}(\mathbb{R}_t; \mathbb{V})} + \nu \|\mathcal{A}u\|_{L^2_{\#}(\mathbb{R}_t; \mathbb{H})} \leq \\ &c (1 + \nu) \|g\|_{H^1_{\#}(\mathbb{R}_t)}. \end{aligned} \tag{6.16}$$

We have used here the periodicity of the solution and (6.11). The estimate for u' follows, in a second stage, directly from the equation (6.15), by appealing to the above estimate for $\mathcal{A}u$. Finally, from these first two estimates, the estimate of the second term on the left hand side follows by appealing to (3.9), with H replaced by $\mathbb{H}$ and so on. Note that $\mathbb{V} = [D(\mathcal{A}), \mathbb{H}]_{1/2}$, independently of (6.5). See [16], Chap.I, Eq. (2.42).

The uniqueness of the solution in the class $L^2_{\#}(\mathbb{R}_t; \mathbb{V})$ follows by setting $f = 0$ in equation (6.12), and by following standard techniques.

Finally, if Ω is regular, (6.5) holds. This can be seen by arguing as in [1]. See also [8], VI.1, Lemma 1.2. A main point here is that the sections Ω_i do not depend on z.

Remark. We may also start by considering problem (6.12) in the truncated domains Θ_r (replace everywhere Θ by Θ_r). This problem admits one and only one solution u_r. For a very elementary proof see, for instance, [22], Chap. 7, problem 7.1-2. One easily verifies that the main estimates do not depend on the parameter r. Denote by u_r the extension by zero of u_r to the whole of Θ. Since the estimates do not depend on r, we may extract an increasing sequence r_n, converging to ∞, and such that the sequence $u_n = u_{r_n}$ converges weakly in $L^2(\mathbb{R}_t; \mathbb{V})$ to some element u. Moreover, u'_n converges weakly to u' in $L^2(\mathbb{R}_t; \mathbb{H})$. Choosing test functions $v \in \mathcal{V}$, passing to the limit in the variational equation as n goes to ∞ and, finally, by appealing to a density argument, we prove that the limit function u satisfies the variational equation for any test function $v \in \mathbb{V}$.

7 The Leray's problem in the Navier-Stokes case ($N \leq 4$)

In the sequel we assume that the dimension $N = n + 1$ satisfies $N \leq 4$. Note that the physical meaningful situation corresponds to $n = 2$. For brevety, we assume here that the Ω_i's are regular.

In the case of the Navier-Stokes equations, we look for solutions v to the following problem:

Problem PLNS: given a real $(2\,\pi)$-time-periodic function $g(t)$ find a $(2\,\pi)$-time-periodic function $v(t, x, z)$ of the Navier-Stokes evolution problem

$$(7.1)\qquad \begin{cases} \frac{\partial v}{\partial t} - \nu\,\Delta\, v + (v \cdot \nabla)\, v + \nabla\, p = 0\,, \\ \nabla \cdot v = 0\,, \quad in \quad \Theta \times \mathbb{R}_t\,; \\ v = 0 \quad on \quad (\partial\,\Theta) \times \mathbb{R}_t\,, \\ v(t + 2\,\pi) = v(t) \quad \forall t \in \mathbb{R}_t\,, \end{cases}$$

such that, for $i = 1,\, 2$,

$$(7.2)\qquad \|v - \chi_i\|_{L^\infty_\#(\mathbb{R}_t; L^2(\Lambda_i))} + \|v - \chi_i\|_{L^2_\#(\mathbb{R}_t; H^1(\Lambda_i))} \leq constant\,.$$

The constraint (7.2) implies convergence of v, in a weak sense, to the χ_i's, as the coordinate z go to infinity. See the end of this section.

Theorem 7.1. *Let $g \in H^1_\#(\mathbb{R}_t)$ be given. There is a positive constant c_0, that depends only on Θ, such that, if* (7.15) *holds, the problem* (7.1), (7.2) *has at least one solution v. The solution v can be written in the form $v = v_0 + u$, where v_0 satisfies* (6.6) *and* (6.7) *and u satisfies* (7.18) *and* (7.4). *In particular, $u = v - \chi_i \quad in \quad \Lambda_i\,, \quad i = 1, 2$, satisfies the asymptotic estimates* (7.20) *and* (7.21).

As in the previous section, we look for solutions v in the form

$$(7.3)\qquad v = v_0 + u\,.$$

Now the problem (6.9) is replaced here by

$$
(7.4)\qquad \begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + (v_0 \cdot \nabla) u + (u \cdot \nabla) v_0 + \nabla p = f(t)\,, \\ \\ \nabla \cdot u = 0\,, \quad in \quad \Theta \times \mathbb{R}_t\,; \\ \\ u = 0 \quad on \quad (\partial \Theta) \times \mathbb{R}_t\,, \\ \\ u(t + 2\pi) = u(t) \quad \forall t \in \mathbb{R}_t\,, \end{cases}
$$

where

$$
f(t) = -\left(\frac{\partial v_0}{\partial t} - \nu \Delta v_0 + (v_0 \cdot \nabla) v_0 \right) + \sum_i \frac{\partial}{\partial z} (z\, \zeta_i(z)\, \psi_i(t))
$$

satisfies (6.10). This last property follows from (2.4), since $(\chi_i \cdot \nabla) \chi_i = 0$.

By appealing to the Sobolev embedding theorem $H^1(\Theta_1) \subset L^4(\Theta_1)$ it readily follows that

$$
\|(v_0 \cdot \nabla) v_0\|^2_{L^2_\#(\mathbb{R}_t; L^2(\Theta_1))} \leq \|v_0\|^2_{L^\infty_\#(\mathbb{R}_t; H^1(\Theta_1))} \|v_0\|^2_{L^2_\#(\mathbb{R}_t; H^2(\Theta_1))}\,.
$$

Hence, by (6.7), one has

$$
(7.5)\qquad \|(v_0 \cdot \nabla) v_0\|^2_{L^2_\#(\mathbb{R}_t; L^2(\Theta_1))} \leq c \sqrt{\nu + \nu^{-4}}\, \|g\|^2_{H^1_\#(\mathbb{R}_t)}\,.
$$

Consequently, (6.13) is replaced here by

$$
(7.6)\qquad \|f\|_{L^2_\#(\mathbb{R}_t; \mathbb{H})} \leq c\,(1 + \nu)\, \|g\|_{H^1_\#(\mathbb{R}_t)} + c \sqrt{\nu + \nu^{-4}}\, \|g\|^2_{H^1_\#(\mathbb{R}_t)}\,.
$$

We look for $u \in L^2_\#(\mathbb{R}_t; \mathbb{V})$ such that

$$
(7.7)\qquad \begin{aligned} & \tfrac{d}{dt}(u, v) + \nu\, ((u, v)) + ((u \cdot \nabla) u, v) + ((v_0 \cdot \nabla) u, v) + \\ & ((u \cdot \nabla) v_0, v) = (f(t), v)\,, \quad \forall v \in \mathbb{V}\,, \end{aligned}
$$

in the distributional sense.

For $N \leq 4$, the proof of the existence of, *at least*, one periodic solution to the problem (7.7) follows well known techniques. The problem can be treated by adapting the classical variational approach, followed in the case of bounded domains, to the domain Θ. The situation is very similar, since Poincaré inequality holds in Θ. One constructs Faedo-Galerkin approximate solutions and shows the existence of the limit solution (possibly non unique). Due to the "extra terms" containing the vector field v_0, one has to assume that the viscosity ν is sufficiently large with respect to the $H^1_\#(\mathbb{R}_t)$ norm of the periodic flux $g(t)$.

Since the technical aspects are well known, we merely present the main formal calculations.

By setting $v = u$ in equation (7.7) one gets

$$
(7.8)\qquad \begin{aligned} & \tfrac{1}{2} \tfrac{d}{dt} \|u\|^2 + \nu \|\nabla u\|^2 \\ & - ((u \cdot \nabla) u, v_0) = (f(t), u)\,. \end{aligned}
$$

By appealing to Hőlder's inequality, to the Sobolev embedding theorem $H^1(\Theta_1) \subset L^4(\Theta_1)$ and to Poincare's inequality one shows that

$$\left| \int_{\Theta_1} (u \cdot \nabla)\, u \,\cdot\, v_0 \, d\overline{x} \right| \leq c\, \|v_0\|_{H^1(\Theta_1)}\, \|\nabla u\|^2_{L^2(\Theta_1)}\,. \tag{7.9}$$

Hence, by (6.7),

$$\left| \int_{\Theta_1} (u \cdot \nabla)\, u \,\cdot\, v_0 \, d\overline{x} \right| \leq c\, \sqrt{\nu + \nu^{-2}}\, \|g\|_{H^1_{\#}(\mathbb{R}_t)}\, \|\nabla u\|^2_{L^2(\Theta_1)}\,. \tag{7.10}$$

On the other hand,

$$\left| \int_{\Lambda^1} (u \cdot \nabla)\, u \,\cdot\, v_0 \, d\overline{x} \right| \leq \int_{z=1}^{+\infty} dz \int_{\Omega} |(u \cdot \nabla)\, u \,\cdot\, v_0|\, dx\,, \tag{7.11}$$

where Λ^1 represents Λ^1_1 or Λ^1_2, and Ω represents Ω_1 or Ω_2. Recall that the sections Ω_i do not depend on z.

Furthermore,

$$\int_{\Omega} |(u \cdot \nabla)\, u \,\cdot\, v_0|\, dx \leq c\, \|v_0\|_{L^4(\Omega)}\, \|\nabla u\|^2_{L^2(\Omega)}\,.$$

By tacking into account that $H^1(\Omega_i) \subset L^4(\Omega_i)$ and that $v_0 = \chi_i$ in Λ_i, and also by appealing to (2.11), it follows that

$$\int_{\Omega} |(u \cdot \nabla)\, u \,\cdot\, v_0|\, dx \leq c\, \sqrt{\nu + \nu^{-1}}\, \|g\|_{H^1_{\#}(\mathbb{R}_t)}\, \|\nabla u\|^2_{L^2(\Omega)}\,,$$

for each t.

Finally, by integration with respect to t, one shows that

$$\left| \int_{\Lambda^1} (u \cdot \nabla)\, u \,\cdot\, v_0 \, dx \right| \leq c\, \sqrt{\nu + \nu^{-1}}\, \|g\|_{H^1_{\#}(\mathbb{R}_t)}\, \|\nabla u\|^2_{L^2(\Lambda^1)}\,. \tag{7.12}$$

From (7.10) and (7.12) one gets

$$|((u \cdot \nabla)\, u,\, v_0)| \leq c_0\, \sqrt{\nu + \nu^{-2}}\, \|g\|_{H^1_{\#}(\mathbb{R}_t)}\, \|\nabla u\|^2_{L^2(\Theta)}\,. \tag{7.13}$$

From (7.8) (7.6) and (7.13) it follows that

$$\frac{1}{2}\,\frac{d}{dt}\|u\|^2 + \nu\, \|\nabla u\|^2 \leq c_0\, \sqrt{\nu + \nu^{-2}}\, \|g\|_{H^1_{\#}(\mathbb{R}_t)}\, \|\nabla u\|^2 + \|f\|\, \|u\|\,. \tag{7.14}$$

Hence, if

$$c_0\, \sqrt{\nu + \nu^{-2}}\, \|g\|_{H^1_{\#}(\mathbb{R}_t)} \leq \frac{\nu}{2} \tag{7.15}$$

then

$$\frac{d}{dt}\|u\|^2 + \nu\, \|\nabla u\|^2 \leq 2\, \|f\|\, \|u\|\,. \tag{7.16}$$

In particular,

$$\frac{d}{dt}\|u\|^2 + c\,\nu\, \|u\|^2 \leq \frac{c_1}{\nu}\, \|f\|^2\,. \tag{7.17}$$

Consequently,

$$\|u(t)\|^2 \leq e^{-c\nu t}\,\|u(0)\|^2 + \frac{c_1}{\nu}\int_0^t e^{-c\nu(t-s)}\,\|f(s)\|^2\,ds\,.$$

It readily follows that the the map $u(0) \to u(2\pi)$ has a fixed point in the ball $B \subset \mathbb{H}$ centered at the origin and with radius

$$\rho = \frac{c_1}{\nu}\frac{\|f\|^2}{1 - \exp\{-2\pi c\,\nu\}}\,.$$

For details see, for instance, [24] page 60 and [22] page 180. If, as in these references, one uses the classical Faedo-Galerkin method, the approximate (periodic) solutions remain inside the ball B. This fact leads to a convergence of a subsequence to a weak solution u of problem (7.4),

$$u \in L^\infty_\#(\mathbb{R}_t; L^2(\Theta)) \cap L^2_\#(\mathbb{R}_t; H^1(\Theta))\,. \tag{7.18}$$

Since

$$v = \chi_i + u \quad in \quad \Lambda_i\,,$$

the solution v of problem (7.1) satisfies (7.2).

The convergence of v to χ_i, in Λ_i, $i = 1, 2$, as z goes to ∞, is equivalent to the convergence of u to zero. As shown below, this last property follows from

$$u \in L^\infty_\#(\mathbb{R}_t; L^2(\Lambda)) \cap L^2_\#(\mathbb{R}_t; H^1(\Lambda))\,. \tag{7.19}$$

ASYMPTOTIC BEHAVIOR. For convenience, we drop in the sequel the index i, $i = 1, 2$, from notations.

We set

$$\Omega(z) = \{(x, z) : x \in \Omega\}\,,$$

and

$$\Lambda_r = \{(x, z) \in \Lambda : z > r\}\,.$$

One has the following result:

Proposition 7.1. *Set* $p = 2/s$ *and let* u *satisfy* (7.19). *Then*

$$\lim_{z\to\infty} \|u\|_{L^p_\#(\mathbb{R}_t; H^{s-\frac{1}{2}}(\Omega_z))} = 0\,, \tag{7.20}$$

for each $s \in [1/2, 1]$. *In particular*

$$\lim_{z\to\infty} \|u\|_{L^p_\#(\mathbb{R}_t; L^q(\Omega_z))} = 0\,, \tag{7.21}$$

where $q = 2\,n/(n + 1 - 2s)$.

Proof. From (7.19) it easily follows, by interpolation, that $u \in L^p_\#(\mathbb{R}_t; H^s(\Lambda))$, for each $s \in [0, 1]$. If $s > 0$ then

$$\lim_{z\to\infty} \|u\|_{L^p_\#(\mathbb{R}_t; H^s(\Lambda_z))} = 0\,. \tag{7.22}$$

Since

$$\|u\|_{H^{s-\frac{1}{2}}(\Omega_z)} \leq c\,\|u\|_{H^s(\Lambda_z)}\,,$$

the thesis follows.

We assume now that $s = 0$ and prove that (7.19) yields

$$\lim_{z\to\infty} \|u\|_{L^4_\#(\mathbb{R}_t;L^2(\Lambda_z))} = 0\,.$$

This estimate can be proved for regular functions and then extended to u by a density argument.

Starting from

$$|u(z,x)|^2 \leq \int_z^{+\infty} |u(s,x)|\,|\frac{\partial u}{\partial z}(s,x)|\,ds\,,$$

one easily shows that

$$\|u\|^2_{L^2(\Omega_z)} \leq \|u\|^2_{L^2(\Lambda_z)}\,\|\nabla\, u\|^2_{L^2(\Lambda_z)}\,,$$

a.e. in $\mathbb{R}_t$. The thesis follows by a straightforward argument.

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