A variational approach to some boundary value problems in the half-line

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Abstract

We study the existence of solutions for two kinds of boundary value problem in the interval $[0, \infty[$. The problems are suggested by models in Mathematical Physics. In the first kind of problem the condition at the left endpoint is $u(0) = \alpha$ while in the second kind a homogeneous Neumann condition $u'(0) = 0$ is imposed. In both cases solutions should satisfy $u(+\infty) = 0$. Our approach is variational, solutions being obtained as minimizers or mountain pass critical points of some functional.

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1 Introduction

There are numerous physical models that motivate the study of boundary value problems in the half-line, from both theoretical and numerical point of view. As significant examples we may cite the existence

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of solitary waves to Klein-Gordon or Schrodinger type equations, the Lagerstrom model for flow and the Thomas-Fermi equation. In fact, a vast literature is available, engaging a diversity of methods ranging from shooting to variational arguments, to fixed-point theorems, to phase-plane techniques. See [1], [2], [3], [5] or [8] for an extensive bibliography.

In this paper we consider two kinds of boundary value problem (b.v.p. for short) for second order differential equations in $[0, +\infty[$. In the first kind of problem the value of the solution at 0 is prescribed, while in the second kind a homogeneous Neumann condition at 0 is imposed. In both cases solutions should vanish at $+\infty$.

A simple, but important example of a problem of the first type is the model (derived independently by Thomas and Fermi in 1927) for determining the electric potential in an isolated neutral atom :

$$u''(t) = \frac{1}{\sqrt{t}}\, u^{\frac{3}{2}} \tag{1.1}$$

$$u(0) = 1 \quad , \quad u(\infty) = 0$$

In fact we shall consider the following more general b.v.p. where $\alpha > 0$ and $p, q \in C(]0, +\infty[)$ and $h \in C([0, +\infty[)$ satisfy conditions to be stated below:

$$u''(t) + p(t)u(t) = q(t)h(u(t)) \tag{1.2}$$

$$u(0) = \alpha \quad , \quad u(+\infty) = 0. \tag{1.3}$$

Existence of a solution to problem (1.1) can be established with undemanding effort with the help of a variational argument. We sketch the idea of the proof. The action functional

$$f(u) = \frac{1}{2}\int_0^\infty u'^2 + \frac{2}{5}\int_0^\infty \frac{1}{\sqrt{t}}\, |u|^{\frac{5}{2}}$$

whose Euler-Lagrange equation is (1.1) is in fact a norm in an obvious reflexive space H. Consider a minimizing sequence in the convex subset $\{u \in H : u(0) = 1\}$; we may suppose that the u_n's are uniformly bounded, positive and decreasing. By passing, if necessary, to a weakly-convergent subsequence assume that $u_n \rightharpoonup u$. The strict convexity of f now implies that u is the unique minimum, necessarily decreasing and $\lim_{t\to\infty} u(t) = 0$ since $f(u) < \infty$.

The relative simplicity of the above reasoning will be detailed and exploited to solve (1.2)-(1.3). We shall use an inequality of Hardy type and well-ordered lower and upper solutions. Although this last method has been extensively used by many authors in bounded domains, it seems to have been much less used in unbounded intervals.

The second kind of b.v.p. studied in this paper may be seen as the problem of finding positive radial solutions of a certain type of semilinear elliptic equation in a ball or in the whole space $\mathbb{R}^N$. The problem that we want to study may be written in the form

$$(t^k u')' = t^k u - t^\alpha u^\beta \tag{1.4}$$

$$u'(0) = 0 \qquad u(M) = 0 \tag{1.5}$$

where M belongs to $]0, \infty]$, $\alpha \geq k > 1$ and $\beta > 1$ ($k = N - 1$ when (1.4) arises from the elliptic equation in N dimensional space). There is a vast bibliography on related problems. For instance, in [3] and [5] solutions are found by topological shooting. In a forthcoming paper, the authors, in collaboration with D. Bonheure have proposed an alternative approach to the results in [3]. That approach provides arguments that can be adapted to a wider class of equations. We work out these ideas in a self contained manner in order to solve (1.4)-(1.5). As a corollary we derive an alternative proof to a result of MacLeod ([5]) relevant in Electromagnetic Theory. We note that the results of [3] are not applicable to (1.4)-(1.5).

We organize this paper as follows: section 2 contains some auxiliary results that will be used in dealing with (1.2)-(1.3) and (1.4)-(1.5); problem (1.2)-(1.3) will be studied in section 3 and problem (1.4)-(1.5) will be studied in section 4.

2 Some auxiliary inequalities

For simplicity, in the sequel, the symbol C will stand for several constants.

Given $\alpha \geq 0$, $p \in C(]0, \infty[)$, let us consider the set of real functions

$$\mathcal{C}_\alpha = \{u : u \text{ is a.c. in } \mathbb{R}^+,\ u(0) = \alpha \text{ and } \int_0^\infty u'(t)^2 dt < \infty\}. \tag{2.6}$$

Lemma 2.1 *Suppose that $p \geq 0$ in $]0, +\infty[$ and there exists a primitive P of p and real numbers ϕ, $\beta \geq 0$ such that*

$P1)$ $P^2(t) \leq \beta p(t)$

$P2)$ $\lim_{t\to 0} P(t)u^2(t) = -\phi \le 0$ for every $u \in \mathcal{C}_\alpha$.

Then

$$(\int_0^\infty pu^2)^{\frac{1}{2}} \le \sqrt{\phi} + 2\sqrt{\beta}\,(\int_0^\infty u'^2)^{\frac{1}{2}} \tag{2.7}$$

for every $u \in \mathcal{C}_\alpha$.

Remark 2.2 *Condition $P2)$ is trivially fulfilled if $P(t)$ is continuous at $t = 0$ ($\phi = |P(0)\alpha^2|$).*

Remark 2.3 *We may take as examples of p's satisfying $P1)$:*

$$p(t) = 1/(t+k)^n,\ n \ge 2$$

or

$$p(t) = \exp(-kt),$$

with $k > 0$.

Proof. Let $u \in \mathcal{C}_\alpha$, *supp* $(u) \subset [0, M]$ for a certain $M > 0$. Integrating by parts we have by $P2)$,

$$\int_0^M p(t)u^2(t)dt = \phi - 2\int_0^M P(t)u(t)u'(t)dt.$$

By the triangular inequality, Cauchy-Schwarz inequality and $P1)$ it follows

$$\int_0^M p(t)u^2(t)dt \le \phi + 2\left(\int_0^M P^2(t)u^2(t)dx\right)^{\frac{1}{2}}\left(\int_0^M u'^2(t)dt\right)^{\frac{1}{2}} \le$$

$$\le \phi + 2\sqrt{\beta}(\int_0^M p(t)u^2(t)dt)^{\frac{1}{2}}(\int_0^M u'^2(t)dt)^{\frac{1}{2}}.$$

Dividing by $(\int_0^M p(t)u^2(t)dt)^{\frac{1}{2}}$ we obtain

$$(\int_0^M p(t)u^2(t)dt)^{\frac{1}{2}} \le \frac{\phi}{(\int_0^M p(t)u^2(t)dt)^{\frac{1}{2}}} + 2\sqrt{\beta}(\int_0^M u'^2(t)dt)^{\frac{1}{2}} \tag{2.8}$$

If $(\int_0^\infty p(t)u^2(t)dt)^{\frac{1}{2}} \le \sqrt{\phi}$ then (2.7) is trivially fulfilled. If not, then (2.8) implies

$$(\int_0^M p(t)u^2(t)dt)^{\frac{1}{2}} \leq \sqrt{\phi} + 2\sqrt{\beta}(\int_0^M u'^2(t)dt)^{\frac{1}{2}}. \tag{2.9}$$

Now, let u be any function in $\mathcal{C}_\alpha$. Given $M > 0$ we define

$$u_{M,\epsilon}(t) = \begin{cases} u(t) & \text{if } t \in [0, M] \\ u(M) + sgn(u(M))\epsilon(M - t) & \text{if } t \in [M, M + |u(M)| / \epsilon] \\ 0 & \text{if } t \in [M + |u(M)| / \epsilon, \infty[\end{cases}$$

(we agree $sgn(0) = 0$). Then, by (2.9)

$$(\int_0^M p(t)u^2(t)dt)^{\frac{1}{2}} \leq (\int_0^\infty p(t)u^2_{M,\epsilon}(t)dt)^{\frac{1}{2}} \leq \sqrt{\phi} + 2\sqrt{\beta}(\int_0^\infty u'^2_{M,\epsilon}(t)dt)^{\frac{1}{2}}.$$

Since

$$\int_0^\infty u'^2_{M,\epsilon}(t)dt = \int_0^M u'^2(t)dt + \epsilon|u(M)|$$

we obtain by letting $\epsilon \to 0$,

$$\left(\int_0^M p(t)u^2(t)dt\right)^{\frac{1}{2}} \leq \sqrt{\phi} + 2\sqrt{\beta}\left(\int_0^M u'^2(t)dt\right)^{\frac{1}{2}}.$$

and the lemma follows from the arbitrariness of M. ■

Remark 2.4 *We obtain Hardy's inequality as a particular case of the previous result. Let u be an absolutely continuous function such that*

$$u(0) = 0, \quad \int_0^\infty u'^2(t)dt < \infty.$$

Then

$$\int_0^\infty u^2(t)/t^2 dt \leq 4\int_0^\infty u'^2(t)dt.$$

In fact, take $\alpha = 0$, $p(t) = \frac{1}{t^2}$, $P(t) = -\frac{1}{t}$. $P1)$ is trivially fulfilled with $\beta = 1$. In order to verify $P2)$ with $\phi = 0$, we show

$$\lim_{t \to 0} \frac{u^2(t)}{t} = 0$$

for any $u \in \mathcal{C}_0$. Since $\int_0^\infty u'^2(t)dt < \infty$, we have $\lim_{t\to 0}\int_0^t u'^2(z)dz = 0$. Therefore, by Cauchy-Schwarz's inequality,

$$\lim_{t\to 0}|u^2(t)/t| = \lim_{t\to 0}|(\int_0^t u'(z)dz)^2/t| \le \lim_{t\to 0}\int_0^t u'^2(z)dz = 0.$$

In the remaining of this section we settle the variational framework to problem (1.4)-(1.5). For the sake of completeness we include sketchy proofs. Given $M \in]0,+\infty[$, we introduce the Hilbert space $H_k(0,M)$ consisting of functions u absolutely continuous in $]0,M]$ such that

$$\|u\|_M^2 := \int_0^M t^k u'(t)^2\, dt \qquad (2.10)$$

is finite and $u(M)=0$. The right-hand side of (2.10) defines (the square of) a norm in this space. We consider also the space $L_\alpha^\upsilon(0,M)$ consisting of measurable functions u such that

$$\int_0^M t^\alpha |u|^\upsilon < \infty.$$

Finally we note $H_k(0,\infty)$ the space of a. c. functions u such that

$$\|u\|^2 = \int_0^\infty t^k u'^2 + \int_0^\infty t^k u^2 < \infty.$$

Lemma 2.5 *Let $u \in H_k(0,M)$ where $M > 0$ and $k > 1$. Then the following inequality holds:*

$$(\int_0^M t^{k-2}u(t)^2\, dt)^{1/2} \le \frac{2}{k-1}(\int_0^M t^k u'(t)^2\, dt)^{1/2}.$$

Proof. Since $\frac{d}{dt}(t^{k-2}u(t)^2) = 2t^{k-2}u(t)u'(t) + (k-2)t^{k-3}u(t)^2$ and $u(M)=0$ we have

$$\int_0^M t^{k-2}u(t)^2\, dt = -2\int_0^M t^{k-1}u(t)u'(t)\, dt - (k-2)\int_0^M t^{k-2}u(t)^2\, dt.$$

Writing $t^{k-1} = t^{(k-2)/2}t^{k/2}$ and using Cauchy-Schwarz for the first integral in the right hand side we obtain

$$(k-1)\int_0^M t^{k-2}u(t)^2\, dt \le 2(\int_0^M t^{k-2}u(t)^2\, dt)^{1/2}(\int_0^M t^k u'(t)^2\, dt)^{1/2}.$$

■

Corollary 2.6 *Let $k > 1$, $M > 0$ and $p > \frac{k-1}{2}$. Then there exists a constant $c = c(M)$ such that for every function $u \in H_k(0, M)$*

$$\|t^p u(t)\|_\infty \le c\|u\|_M.$$

Proof. We have $t^p u(t) = -\int_t^M (pt^{p-1}u(t) + t^p u'(t))\, dt$. Since $2p - k > -1$ we write

$$t^{p-1}u(t) = t^{p-k/2}(t^{k/2-1}u(t)), \quad t^p u'(t) = t^{p-k/2}(t^{k/2}u'(t))$$

as products of two functions in $L^2(0, M)$ and apply Cauchy-Schwarz and lemma 2.5 to conclude. ∎

Lemma 2.7 *Let $k, \beta > 1$, $\alpha > \frac{(k-1)(\beta+1)}{2} - 1$. Then there exists $C > 0$ such that for every $u \in H_k(0, M)$*

$$\int_0^M t^\alpha |u|^{\beta+1} \le C\|u\|_M^{\beta+1} \tag{2.11}$$

Moreover $H_k(0, M)$ is compactly embedded in $L_\alpha^{\beta+1}(0, M)$.

Proof. We have

$$\int_0^M t^\alpha \, |u|^{\beta+1} \le \left\| t^{(\alpha-k+2)/(\beta-1)} \, |u| \right\|_\infty^{\beta-1} \int_0^M t^{k-2} \, |u|^2 \, .$$

Our assumptions on α imply $(\alpha - k + 2)/(\beta - 1) > \frac{k-1}{2}$. Then by lemma 2.5 and corollary 2.6 we have

$$\int_0^M t^\alpha \, |u|^{\beta+1} \le C\|u\|_M{}^{\beta+1}.$$

In order to prove the final assertion we note that, for any $\epsilon > 0$, we have a compact embedding of $H_k(0, M)$ in $C[\epsilon, M]$. Choose $\delta > 0$ such that $\alpha - \delta > \frac{(k-1)(\beta+1)}{2} - 1$. Then,

$$\int_0^\epsilon t^\alpha \, |u|^{\beta+1} \le \epsilon^\delta \int_0^\epsilon t^{\alpha-\delta} \, |u|^{\beta+1} \le \; C\epsilon^\delta \, \|u\|_M^{\beta+1}$$

and the conclusion now follows from standard arguments. ∎

Lemma 2.8 *Assume* $\frac{(k-1)(\beta+1)}{2} - 1 < \alpha \leq \frac{k(\beta+1)}{2}$. *Then there exists* C *such that, for any* $u \in H_k(0,\infty)$

$$\int_0^\infty t^\alpha |u(t)|^{\beta+1}\, dt \leq C \|u\|^{\beta+1}.$$

Proof. We have

$$\int_1^\infty t^\alpha |u(t)|^{\beta+1}\, dt \leq \left\| t^{(\alpha-k)/(\beta-1)} |u| \right\|_{L_\infty(1,\infty)}^{(\beta-1)} \|u\|^2 .$$

We note that our assumptions on α imply that $(\alpha-k)/(\beta-1) \leq k/2$. Trivially $t^{k/2}u \in H([1,\infty[)$ and, since $H([1,\infty[)$ is continuously embedded in $C([1,\infty[)$, we conclude:

$$\left\| t^{(\alpha-k)/(\beta-1)} |u| \right\|_{L_\infty(1,\infty)} \leq C \left\| t^{k/2} u \right\|_{H_1([1,\infty[)} \leq C \|u\| . \tag{2.12}$$

Therefore,

$$\int_1^\infty t^\alpha |u(t)|^{\beta+1}\, dt \leq C \|u\|^{\beta+1}. \tag{2.13}$$

For the remaining part of the integral we write

$$\int_0^1 t^\alpha |u(t)|^{\beta+1}\, dt \leq C \left(\int_0^1 t^\alpha |u(t)-u(1)|^{\beta+1}\, dt + \int_0^1 t^\alpha |u(1)|^{\beta+1}\, dt \right).$$

By lemma 2.7, we have

$$\int_0^1 t^\alpha |u(t)-u(1)|^{\beta+1}\, dt \leq C \|u-u(1)\|_1{}^{\beta+1}$$

and by (2.12)

$$\int_0^1 t^\alpha |u(1)|^{\beta+1}\, dt = (\alpha+1)^{-1} |u(1)|^{\beta+1} \leq C \|u\|^{\beta+1}.$$

Therefore we obtain

$$\int_0^1 t^\alpha |u(t)|^{\beta+1}\, dt \leq C \|u\|^{\beta+1}. \tag{2.14}$$

Finally (2.13) together with (2.14) imply the assertion of the lemma. ■

Corollary 2.9 *$H_k(0,\infty)$ is compactly embedded in $L_\alpha^{\beta+1}(0,\infty)$ provided that $\frac{(k-1)(\beta+1)}{2} - 1 < \alpha < \frac{k(\beta+1)}{2}$.*

Proof. Let $u_n \to 0$ weakly in $H_k(0,\infty)$. We may choose δ such that

$$\frac{(k-1)(\beta+1)}{2} - 1 < \alpha \pm \delta < \frac{k(\beta+1)}{2}.$$

By lemma 2.8, we have, for every $\epsilon > 0$,

$$\int_0^\epsilon t^\alpha \left|u_n(t)\right|^{\beta+1} dt \leq \epsilon^\delta \int_0^\epsilon t^{\alpha-\delta} \left|u_n(t)\right|^{\beta+1} \leq \epsilon^\delta \left\|u_n\right\|^{\beta+1}.$$

Also, for $M > 1$,

$$\int_M^\infty t^\alpha \left|u_n(t)\right|^{\beta+1} dt \leq M^{-\delta} \int_M^\infty t^{\alpha+\delta} \left|u_n(t)\right|^{\beta+1} \leq M^{-\delta} \left\|u_n\right\|^{q+1}.$$

The proof now follows from the arbitrariness of ϵ and M and the L_∞ convergence of (u_n) in compact subintervals of $]0,\infty[$. ■

Proposition 2.10 *Let $M > 0$, $\frac{(k-1)(\beta+1)}{2} - 1 < \alpha$. Then:*

(i) the functional defined in $H_k(0,M)$ as

$$J_M(u) = \int_0^M \frac{1}{2}[(t^k u'^2 + t^k (u^+)^2) - \frac{1}{\beta+1} t^\alpha (u^+)^{\beta+1}]dt$$

is of class C^1 and satisfies Palais-Smale (PS).

(ii) A nontrivial critical point of J_M belongs to $C^1([0,M]) \cap C^2(]0,M])$ and is a positive solution of (1.4)-(1.5).

Proof. Using lemma 2.7 one proves via standard arguments that J_M is of class C^1 in $H_k(0,M)$. In addition $J'(u) = u + N(u)$ where $N : H_k(0,M) \to H_k(0,M)$ is completely continuous. Let u_m be a Palais-Smale sequence, i.e. such that $J(u_m)$ is bounded and $J'(u_m) \to 0$. According to a standard reasoning (see [7]), it suffices to show that u_m is bounded. We have for sufficiently large m

$$J(u_m) = \int_0^M [\frac{1}{2}(t^k {u_m}'^2 + t^k ({u_m}^+)^2) - \frac{1}{\beta+1} t^\alpha ({u_m}^+)^{\beta+1}]dt \leq C, \quad (2.15)$$

and

$$\langle J'(u_m), u_m\rangle = \int_0^M t^k {u_m'}^2 + t^k({u_m}^+)^2 - t^\alpha ({u_m}^+)^{\beta+1} dt \geq -\|u\|_M. \tag{2.16}$$

Multiplying (2.16) by $\frac{1}{\beta+1}$ and subtracting it from (2.15) we obtain

$$(\frac{1}{2} - \frac{1}{\beta+1})\|u\|_M{}^2 \leq C + \frac{1}{\beta+1}\|u\|_M,$$

and since $\beta > 1$ we conclude that the u_m's are uniformly bounded.

In order to prove (ii), suppose that u is a nontrivial critical point of J_M. We must verify that u is positive. Since

$$(t^k u'(t))' = t^k u^+ - t^\alpha (u^+)^\beta \tag{2.17}$$

in $]0, M[$ and $u(M) = 0$ there exists necessarily t_0 such that $u(t_0) > 0$. If, for a certain $0 < t_1 < t_0$, $u(t_1) = 0$ and $u'(t_1) = 0$ the existence and uniqueness theorem would imply $u \equiv 0$. If $u'(t_1) > 0$ by (2.17)

$$u'(t) = u'(t_1)t_1^k/t^k$$

for all $0 < t \leq t_1$, but this is impossible since $\|u\|_M < \infty$. If for a certain $t_2 > t_0$, $u(t_2) = 0$ we conclude with similar arguments from (2.17) and the existence uniqueness theorem that $u(M) < 0$. We may therefore state that $u(t) > 0$ for all $t \in]0, M[$ and u satisfies (1.4) in $]0, M[$. It remains to prove that the boundary condition $u'(0) = 0$ is fulfilled. Integrating by parts between t_1 and t_2 we get by Holder's inequality,

$$\left|t_2^k u'(t_2) - t_1^k u'(t_1)\right| = \left|\int_{t_1}^{t_2} t^k u - t^\alpha u^\beta dt\right| \leq$$

$$\leq \left(\int_{t_1}^{t_2} t^k\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} t^k u^2\right)^{\frac{1}{2}} + \left(\int_{t_1}^{t_2} t^\alpha\right)^{1/(\beta+1)} \left(\int_{t_1}^{t_2} t^\alpha u^{\beta+1}\right)^{\beta/(\beta+1)}.$$

By lemma 2.7 this trivially implies $t^k u'(t)$ has a limit as $t \to 0$ and this limit must be zero since $u \in H_k(0, M)$. Hence

$$\left|t^k u'(t)\right| \leq o(1)(t^{(\alpha+1)/(\beta+1)} + t^{(k+1)/2})$$

or

$$\left|u'(t)\right| \leq o(1)(t^{(\alpha+1)/(\beta+1)-k} + t^{(-k+1)/2})$$

Now, given an estimate $|u'(t)| \leq ct^\rho$, we obtain, integrating u' between t and a fixed t_1:

$$|u(t)| \leq C + Ct^{\rho+1}.$$

Since u satisfies (1.4) we obtain the following estimates

$$|t^k u'(t)| \leq C[\int_0^t s^k(1+s^{\rho+1}) + \int_0^t s^\alpha(1+s^{\rho+1})^\beta]$$

or

$$|u'(t)| \leq C(t + t^{\rho+2} + t^{\alpha+1-k} + t^{\beta(\rho+1)+\alpha+1-k}).$$

Let us consider the following recurrence relation:

$$\rho_0 = \min\{(\alpha+1)/(\beta+1) - k, (-k+1)/2\}$$

$$\rho_{n+1} = \beta(\rho_n + 1) + \alpha + 1 - k$$

Since $\beta > 1$, the sequence ρ_n will tend to $+\infty$ provided that $\rho_0 > x_0$, where $x_0 = (\beta + \alpha + 1 - k)/(1-\beta)$ is the abscissa of the intersection point of the lines $y = x$ and $y = \beta(x+1) + \alpha + 1 - k$. Solving explicitly this inequality in α one gets

$$\alpha > \max\{\frac{(k-1)(\beta+1)}{2} - 1, \frac{(k-1)(\beta+1)}{2} - \beta\}$$

but since $\beta > 1$ one retains

$$\alpha > \frac{(k-1)(\beta+1)}{2} - 1$$

which is our assumption on α. ∎

3 Problem (1.2)-(1.3)

We recall that we are interested in the boundary value problem

$$u''(t) + p(t)u(t) = q(t)h(u(t)) \tag{1.2}$$

$$u(0) = \alpha, \quad u(+\infty) = 0 \tag{1.3}$$

with $\alpha > 0$. We assume that $p, q \in C(]0, +\infty[)$ and $h \in C([0, +\infty[)$. We introduce the following Hilbert space

$$E = \{u : u \in H^1_{loc}(0,\infty)\,, \int_0^\infty u'(t)^2 dt + \int_0^\infty p(t)u^2(t)dt < \infty\}.$$

Theorem 3.1 *Suppose that* $p \geq 0$ *and there exists a primitive* P *of* p *such that* $P1)$ *and* $P2)$ *hold with* $0 \leq \beta < \frac{1}{4}$. *Moreover assume*

A) $q > 0$, $q \in L^1(0,1)$ *and* $\int_1^\infty q = +\infty$,

B) $h(0) = 0$ *and* $h(u) > 0$ *for all* $u > 0$.

C) $\lim_{t\to+\infty} \frac{p(t)}{q(t)} = 0$,

D) $\liminf_{u\to+\infty} \frac{h(u)}{u} > 0$.

Then problem (1.2)-(1.3) has a nonnegative solution that minimizes the action functional $f : \mathcal{C}_{\alpha,E} \mapsto \mathbb{R} \cup \{+\infty\}$,

$$f(u) = \frac{1}{2}\int_0^\infty u'(t)^2 dt - \frac{1}{2}\int_0^\infty p(t)u^2(t)dt + \int_0^\infty q(t)H(u(t))dt$$

where $\mathcal{C}_{\alpha,E} = \mathcal{C}_\alpha \cap E = \{u \in E : u(0) = \alpha\}$ *and* $H(u) = \int_0^u h(z)dz$ *for* $u \geq 0$.

Proof. Extend h to $]-\infty, 0]$ with value 0. Usual arguments prove that if $\underline{u}$ is a nonnegative minimizer of f with the new h in $\mathcal{C}_{\alpha,E}$ then it is a classical solution of (1.2) with the original h. We will prove that the functional f attains a minimum $\underline{u}$ in $\mathcal{C}_{\alpha,E}$.

Since $q,\ H \geq 0$, we have, by lemma 2.1,

$$f(u) \geq \frac{1}{2}\int_0^\infty u'(t)^2 dt - \frac{1}{2}\int_0^\infty p(t)u^2(t)dt \geq$$

$$\left(\frac{1}{2} - 2\beta\right)\int_0^\infty u'(t)^2 dt - 2\sqrt{\beta\phi}\left(\int_0^\infty u'(t)^2 dt\right)^{\frac{1}{2}} - \phi/2, \qquad (3.18)$$

and the assumption $\beta < \frac{1}{4}$ implies that the functional is bounded from below. Let (u_n) be a minimizing sequence. Then, by (3.18) we have

$$\int_0^\infty u_n'(x)^2 \leq M_1$$

for a certain $M_1 \geq 0$. Together with (2.7) this implies

$$\int_0^\infty p(t)u_n^2(t)dt \leq M_2$$

for a certain $M_2 \geq 0$.

The above two inequalities imply that (u_n) is bounded. Hence we may suppose that $(u_n) \rightharpoonup \underline{u}$ in E. Moreover $\mathcal{C}_{\alpha,E}$ is closed and convex therefore, by Mazur's theorem, $\underline{u} \in \mathcal{C}_{\alpha,E}$.

We must verify that f is w.l.s.c. (weakly lower semi-continuous). Let (u_n) be a bounded sequence in $\mathcal{C}_{\alpha,E}$. By compact imbedding in finite intervals, we may suppose that u_n converges pointwise in $[0,+\infty[$ to some function $\underline{u} \in E$. Since $q(t)H(u) \geq 0$ we can apply Fatou's lemma to the nonlinear term in the functional and obtain:

$$\liminf_{n\to\infty} \int_0^\infty q(t)H(u_n(t))dt \geq \int_0^\infty q(t)H(\underline{u}(t))dt \tag{3.19}$$

By lemma 2.1 the quadratic form

$$\langle u,u\rangle = \left(\frac{1}{2}\int_0^\infty u'(t)^2 dt - \frac{1}{2}\int_0^\infty p(t)u^2(t)dt\right)$$

is bounded from below in $\mathcal{C}_{\alpha,E}$, say by C. Let $u_1, u_2 \in \mathcal{C}_{\alpha,E}$. Then, since for any real λ, $(1-\lambda)u_1 + \lambda u_2 \in \mathcal{C}_{\alpha,E}$, we have

$$\Gamma(\lambda) = \big\langle (1-\lambda)u_1 + \lambda u_2, (1-\lambda)u_1 + \lambda u_2\big\rangle \geq C.$$

Consequently $\Gamma(\lambda)$ is a quadratic polynomial bounded from below therefore necessarily convex. Then, for any λ in $[0,1]$,

$$\Gamma(\lambda) \leq \lambda\Gamma(0) + (1-\lambda)\Gamma(1)$$

or equivalently, if $0 \leq \lambda \leq 1$

$$\big\langle (1-\lambda)u_1 + \lambda u_2, (1-\lambda)u_1 + \lambda u_2\big\rangle \leq (1-\lambda)\langle u_1,u_1\rangle + \lambda\langle u_2,u_2\rangle$$

The convexity and continuity of $\langle .\,,.\rangle$ in $\mathcal{C}_{\alpha,E}$ imply that it is w.l.s.c.. Together with (3.19) it allows us to conclude that f is w.l.s.c.. Then $\liminf_{n\to\infty} f(u_n) \geq f(\underline{u})$ and $\underline{u}$ is a minimum of f. Of course $\underline{u}(0) = \alpha$. It remains to prove that $\underline{u} \geq 0$ and

$$\lim_{t\to\infty} \underline{u}(t) = 0.$$

Suppose that $\underline{u} < 0$ in $]a,b[$ with $a > 0$, $\underline{u}(a) = 0$. Then $\underline{u}'' + p(t)\underline{u} = 0$ in $[a,b[$ and $\underline{u}$ is convex in $[a,b[$. If $b = +\infty$ then $\underline{u}'(+\infty) = 0$ (since $\underline{u} \in E$), $\int_a^\infty (\underline{u}'^2 - p(t)\underline{u}^2)\,dt = \lim_{t\to+\infty} \underline{u}(t)\underline{u}'(t) \geq 0$. Hence the function $v(t) = \underline{u}(t)$ if $t \leq a$, $v(t) = 0$ if $t \geq a$ would also be a minimizer of f. A similar and simpler argument allows us to replace $\underline{u}$ with 0 in $]a,b[$ if $b < +\infty$ and $\underline{u}(b) = 0$.

Suppose that $\liminf_{t\to\infty} \underline{u}(t) > 0$. Then, by our assumptions on h, we would have for a certain $L > 0$, $H(u(t)) \geq \epsilon > 0$ for all $t \geq L$. Consequently, by A)

$$\int_0^\infty q(t)H(\underline{u}(t))dt = \infty, \tag{3.20}$$

a contradiction. Therefore

$$\liminf_{t\to\infty} \underline{u}(t) = 0.$$

Now, suppose that

$$\limsup_{t\to\infty} \underline{u}(t) > 0.$$

Then there exists $\delta > 0$ and a sequence $t_n \to +\infty$ so that $\underline{u}(t_n) \geq \delta$ is a local maximum of $\underline{u}$. Since $\underline{u}$ satisfies (1.2)

$$p(t_n)\underline{u}(t_n) \geq q(t_n)h(\underline{u}(t_n))$$

which clearly contradicts B), C) and D).

Remark 3.2 *If $h(u)$ is locally Lipchitz it is easy to see that the solution $\underline{u}$ thus obtained satisfies $\underline{u}(t) > 0$, $\forall t \geq 0$.*

For simplicity, in the next result, we consider $q \equiv 1$ and obtain a solution via a minimization method in the presence of well ordered lower and upper solutions.

Theorem 3.3 *Suppose that $q \equiv 1$,*

$P3$) $p \in C(]0,\infty[,\mathbb{R}^+) \cap L^1(]0,\infty[) \cap L^r(]0,\infty[)$, $r > 1$

$H1$) h is locally Lipchitz, $H(u) = \int_0^u h(z)dz \geq C|u|^{2r'}$ where $\frac{1}{r'} + \frac{1}{r} = 1$ and $C > 0$.

$S1$) There exists $s \in C^2([0,\infty[,\mathbb{R}^+)$ such that $|s|_\infty \leq K < \infty$, $s(0) > \alpha$ and $s''(t) + p(t)s(t) - h(s(t)) < 0$.

Then problem (1.2)-(1.3) has a solution.

Remark 3.4 *The following proof shows that the theorem holds if we take q continuous and $c_1 \leq q(t) \leq c_2$ where $c_1, c_2 > 0$.*

Proof. As in theorem 3.1 we consider the space E and the functional f already defined. Set

$$\mathcal{C} = \{u \in E : u(0) = \alpha,\ 0 \leq u(t) \leq s(t) \text{ for all } t \in [0,\infty[\}.$$

We show that f is coercive in $\mathcal{C}$. By Cauchy-Schwarz's inequality and assumption $P3)$ we have

$$\int_0^\infty u'^2 + \int_0^\infty pu^2 \le \int_0^\infty u'^2 + \left(\int_0^\infty p^r\right)^{\frac{1}{r}} \left(\int_0^\infty |u|^{2r'}\right)^{\frac{1}{r'}} \le$$

$$\le \int_0^\infty u'^2 + C^{-\frac{1}{r'}} \left(\int_0^\infty p^r\right)^{\frac{1}{r}} \left(\int_0^\infty H(u)\right)^{\frac{1}{r'}}.$$

Suppose that (u_n) is a sequence in $\mathcal{C}$ such that $\int_0^\infty (u_n')^2 + \int_0^\infty pu_n^2 \to \infty$; then

$$\int_0^\infty (u_n')^2 + \int_0^\infty H(u_n) \to \infty,$$

and, since

$$f(u_n) \ge \frac{1}{2} \int_0^\infty u'^2 - \frac{K}{2} \|p\|_{L_1} + \int_0^\infty H(u_n),$$

we conclude $f(u_n) \to \infty$.

Let (u_n) be a minimizing sequence, necessarily bounded by the above estimates. We extract a weakly convergent subsequence still denoted by (u_n) and define $\underline{u}$ to be its limit. The convexity of $u \mapsto \int_0^\infty u'^2$ and Fatou's lemma now imply that

$$\int_0^\infty \underline{u}'^2 + \int_0^\infty H(\underline{u}) \quad \le \quad \lim_{n\to\infty} \int_0^\infty (u_n')^2 + \int_0^\infty H(u_n).$$

We prove

$$\lim_{n\to\infty} \int_0^\infty pu_n^2 = \int_0^\infty p\underline{u}^2.$$

By $P3)$ and the fact that $\|u_n\|_\infty \le K$ for every $n \in \mathbb{N}$, we have that given ϵ, there exists $M > 0$ such that

$$\int_M^\infty pu_n^2 < \frac{\epsilon}{2}$$

for all $n \in \mathbb{N}$. Clearly a weakly convergent sequence in E converges in $C[0, M]$, therefore there exists an order l such that $n > l$ implies

$$\left| \int_0^M pu_n^2 - \int_0^M p\underline{u}^2 \right| < \frac{\epsilon}{2},$$

and the assertion follows.

Let us prove that $\underline{u}$ is a solution to problem (1.2)-(1.3). If $0 < \underline{u}(t) < s(t)$ for all $t \neq 0$ then for all $h \in C^1_c\,(]0,\infty[)$ we have, for λ sufficiently small,

$$\underline{u} + \lambda h \in \mathcal{C}$$

Consequently $\underline{u}$ is an extremal of f in E and the assertion follows. In order to prove that $u > 0$ we argue by contradiction. Suppose that for a certain $t_0 > 0$, $\underline{u}(t_0) = 0$ and $t_0 = \min\{t : \underline{u}(t) = 0\}$. Then, since $\underline{u}$ is nontrivial , the existence and uniqueness theorem implies $\underline{u}'(t) \leq -\gamma < 0$ in an interval $]t_0 - \nu, t_0]$. We consider

$$g_\epsilon(t) = (\epsilon - |t - t_0|)^+ \tag{3.21}$$

For $\epsilon < \nu$ one verifies easily that

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \underline{u}' g_\epsilon' = \int_{t_0-\epsilon}^{t_0} \underline{u}' - \int_{t_0}^{t_0+\epsilon} \underline{u}' \leq$$

$$\leq \int_{t_0-\epsilon}^{t_0} \underline{u}' - (\underline{u}(t_0+\epsilon) - \underline{u}(t_0)) \leq\, -\gamma\epsilon.$$

Moreover

$$\left|\int_{t_0-\epsilon}^{t_0+\epsilon} p\underline{u} g_\epsilon\right| + \left|\int_{t_0-\epsilon}^{t_0+\epsilon} h(\underline{u}) g_\epsilon\right| \leq C\epsilon^2.$$

Then for sufficiently small ϵ

$$\langle f'(u), g_\epsilon\rangle = \int_{t_0-\epsilon}^{t_0+\epsilon} \underline{u}' g_\epsilon' - \int_{t_0-\epsilon}^{t_0+\epsilon} p\underline{u} g_\epsilon + \int_{t_0-\epsilon}^{t_0+\epsilon} h(\underline{u}) g_\epsilon < 0,$$

a contradiction since the fact that $\underline{u}$ is a minimizer in $\mathcal{C}$ necessarily implies

$$\lim_{l\to 0^+} \left(f(\underline{u} + l g_\epsilon) - f(\underline{u})\right)/l \geq 0.$$

It remains to prove that $\underline{u}(t) < s(t)$ for all $t \in [0,\infty[$. Suppose that $\underline{u}(t_0) = s(t_0)$ for a certain $t_0 > 0$ (again we may suppose that t_0 is the minimum of the t's having this property). By $(S1)$ we have, for a fixed ϵ_0, a certain $\delta > 0$ such that

$$s''(t) + p(t)s(t) - h(s(t)) \leq -\delta < 0$$

for all $t \in [t_0 - \epsilon_0, t_0 + \epsilon_0]$. Then, considering g_ϵ as in (3.21) with $\epsilon < \epsilon_0$ we obtain, integrating by parts,

$$\langle f'(\underline{u}), -g_\epsilon \rangle = -\int_{t_0-\epsilon}^{t_0+\epsilon} \underline{u}' g_\epsilon' + \int_{t_0-\epsilon}^{t_0+\epsilon} p\underline{u} g_\epsilon - \int_{t_0-\epsilon}^{t_0+\epsilon} h(\underline{u}) g_\epsilon =$$

$$\int_{t_0-\epsilon}^{t_0+\epsilon} (s' - \underline{u}')\, g_\epsilon' + \int_{t_0-\epsilon}^{t_0+\epsilon} p(\underline{u} - s) g_\epsilon - \int_{t_0-\epsilon}^{t_0+\epsilon} (h(\underline{u}) - h(s)) g_\epsilon - \int_{t_0-\epsilon}^{t_0+\epsilon} s' g_\epsilon' +$$
$$\int_{t_0-\epsilon}^{t_0+\epsilon} p s g_\epsilon - \int_{t_0-\epsilon}^{t_0+\epsilon} h(s) g_\epsilon \le$$

$$\underline{u}(t_0 - \epsilon) - s(t_0 - \epsilon) + \underline{u}(t_0 + \epsilon) - s(t_0 + \epsilon) +$$
$$\int_{t_0-\epsilon}^{t_0+\epsilon} p(\underline{u} - s) g_\epsilon - \int_{t_0-\epsilon}^{t_0+\epsilon} (h(\underline{u}) - h(s)) g_\epsilon +$$
$$\int_{t_0-\epsilon}^{t_0+\epsilon} (s''(t) + p(t)s(t) - h(s(t))) g_\epsilon dt \le$$

$$\int_{t_0-\epsilon}^{t_0+\epsilon} p(\underline{u} - s) g_\epsilon - \int_{t_0-\epsilon}^{t_0+\epsilon} (h(\underline{u}) - h(s)) g_\epsilon - \int_{t_0-\epsilon}^{t_0+\epsilon} \delta g_\epsilon dt.$$

By the continuity of $\underline{u}, s, h, p$ we may suppose that $|p(\underline{u} - s) - (h(\underline{u}) - h(s))| < \delta/2$ a.e in $[t_0 - \epsilon, t_0 + \epsilon]$, for sufficiently small ϵ. Then

$$\langle f'(\underline{u}), -g_\epsilon \rangle < -\frac{1}{2} \int_{t_0-\epsilon}^{t_0+\epsilon} \delta g_\epsilon dt < 0,$$

a contradiction. Finally, we show that $\lim_{t\to\infty} \underline{u}(t) = 0$.

Suppose that $\liminf_{t\to\infty} |\underline{u}(t)| > 0$. Then, by our assumptions on h, we would have for a certain $L > 0$, $H(u(t)) \ge \epsilon > 0$ for all $t \ge L$. Consequently,

$$\int_0^\infty H(\underline{u}(t)) dt = \infty, \tag{3.22}$$

a contradiction. Therefore

$$\liminf_{t\to\infty} |\underline{u}(t)| = 0.$$

Suppose that

$$\limsup_{t\to\infty} |\underline{u}(t)| > 0.$$

By $H1)$ there exist sufficiently small $\delta, \rho > 0$ and a sequence (t_n) such that $H(\underline{u}(t_{2n})) = \rho/2$, $H(\underline{u}(t_{2n+1})) = \rho$, $|\underline{u}(t_{2n+1}) - \underline{u}(t_{2n})| = \delta$ and $H(\underline{u}(t)) \geq \rho/2$ for all $t \in [t_{2n}, t_{2n+1}]$. Since

$$\delta = |\underline{u}(t_{2n+1}) - \underline{u}(t_{2n})| = \left| \int_{t_{2n}}^{t_{2n+1}} \underline{u}'(z)dz \right| \leq \sqrt{\int_{t_{2n}}^{t_{2n+1}} \underline{u}'^2(z)dz} \sqrt{t_{2n+1} - t_{2n}} \leq$$

$$\leq \sqrt{\int_0^\infty \underline{u}'^2(z)dz} \sqrt{t_{2n+1} - t_{2n}},$$

we have

$$|t_{2n+1} - t_{2n}| \geq \delta^2/K^2,$$

with $K = \sqrt{\int_0^{\infty_1} \underline{u}'^2(z)dz}$. Then, for every $n \in \mathbb{N}$,

$$\int_0^\infty H(\underline{u}(t))dt \geq \sum_{j=1}^n \int_{t_{2j}}^{t_{2j+1}} H(\underline{u}(t))dt \geq n(\delta^2\rho/2K^2)$$

and we obtain the same contradiction as in (3.22).

■

4 Problem (1.4)-(1.5)

We recall that we are interested in the problem

$$(t^k u')' = t^k u - t^\alpha u^\beta \tag{1.4}$$

$$u'(0) = 0 \quad u(M) = 0. \tag{1.5}$$

We denote by (P_M) the problem (1.4)-(1.5) when M is prescribed. We state the main result of this section:

Theorem 4.1 *Assume $\alpha \geq k > 1$, $\beta > 1$ and $\frac{(k-1)(\beta+1)}{2} - 1 < \alpha < \frac{k(\beta+1)}{2}$. Then, there exists M_0 such that if $M \in]M_0, \infty]$ then problem (P_M) has at least one solution for every $M \in]M_0, \infty]$.*

We consider first the existence of positive solutions to problem (P_M) when M is finite. In what follows we shall make use of a function ϕ_b defined by

$$\phi_b(t) = \begin{cases} \xi & t \in [0,b], \\ \xi + b - t & t \in [b, b+\xi], \\ 0 & \text{otherwise.} \end{cases} \tag{4.23}$$

Here ξ is such that $\xi - \xi^\beta < 0$. If $\alpha \geq k$ and $\beta > 1$ easy estimates imply the existence of sufficiently large b such that

$$J_M(\phi_b) < 0$$

for all $M > \xi + b$.

Lemma 4.2 *Suppose that for a certain $M_1 > 0$ there exists $\phi \in H_k(0, M_1)$ such that $J_{M_1}(\phi) < 0$. Then problem (P_M) has a nontrivial positive solution for every $M \geq M_1$.*

Proof. Extending ϕ in $[M_1, \infty[$ by zero we have

$$J_M(\phi) < 0,$$

for every $M > M_1$. Moreover J_M has a strict local minimum at the origin. In fact, for an appropriate constant $C > 0$, we have by lemma 2.7

$$\int_0^M t^\alpha (u(t)^+)^{\beta+1} \geq -C\|u\|_M{}^{\beta+1}$$

therefore

$$J_M(u) \geq \|u\|_M{}^2 - C\|u\|_M{}^{\beta+1}.$$

Proposition 2.10 now implies the existence of a nontrivial solution which is a critical point of mountain pass type (see [7]). ■

<u>Proof of theorem 4.1</u>: We may take M_0 as the infimum of the M's such that there exists $\phi \in H_k(0, M)$ with $J_M(\phi) < 0$. By the previous lemma we have a solution to P_M whenever $M_0 < M < \infty$. In order to prove that P_∞ has a solution we will consider an approximating procedure.

Take a sequence of values $M \to \infty$ and according to the preceding lemma there is a corresponding sequence u_M of nonnegative functions in $H_k(0, \infty)$ obtained as the extension by zero of critical points of Mountain Pass type in $H_k(0, M)$. By considering the segment that connects the origin to the function ϕ_b in a fixed $H_k(0,\bar{M})$ the inf-max characterization of the corresponding critical value implies

$$0 < J_M(u_M) < d$$

where d can be fixed independently of M. Also $\langle J'_M(u_M), u_M\rangle = 0$. Therefore,

$$\int_0^M (\frac{1}{2}t^k(u'_M)^2 + \frac{1}{2}t^k u_M^2 - \frac{1}{\beta+1}t^\alpha u_M^{\beta+1})dt \leq d \qquad (4.24)$$

$$\int_0^M (t^k(u'_M)^2 + t^k u_M^2 - t^\alpha u_M^{\beta+1})dt = 0 \qquad (4.25)$$

Multiplying this equality by $\frac{1}{\beta+1}$ and subtracting it to (4.24) we get

$$(\frac{1}{2} - \frac{1}{\beta+1})\int_0^M t^k(u'_M)^2 + t^k u_M^2 dt \leq d$$

or

$$\|u_M\| \leq d_1$$

We also have by (4.25) and lemma (2.1)

$$\|u_M\| \leq d_2 \|u_M\|^{\beta+1}$$

and since the u_M's are nontrivial,

$$\|u_M\| \geq d_2^{\frac{1}{\beta-1}}. \qquad (4.26)$$

Therefore along some subsequence of values $M \to \infty$

$$u_M \rightharpoonup v \quad \text{in} \quad H_k(0,\infty).$$

On the other hand, we deduce from (4.25), (4.26) and corollary 2.9 that

$$\int_0^\infty t^k(u'_M)^2 + t^k u_M^2 dt = \int_0^\infty t^\alpha u_M^{\beta+1}dt \to \int_0^\infty t^\alpha v^{\beta+1}dt > 0.$$

It follows that $v \neq 0$. By compactness in bounded intervals v is a solution to equation (1.4) and repeating the same arguments of proposition 2.10 we obtain $C > 0$ such that:

$$|v'(t)| \leq Ct,$$

for small values of t. Therefore v is a solution to (P_∞). ■

Example. A simple application of theorem 4.1 provides a new existence proof to a boundary value problem arising in Electromagnetic Theory that has been considered by MacLeod in [5]:

$$y'' + \frac{2}{t}y' + \Big[y - (1 + \frac{2}{t^2})\Big]y = 0 \tag{4.27}$$

$$y(0) = 0 \qquad y(\infty) = 0 \tag{4.28}$$

To see that this problem has a nontrivial positive solution we take the new dependent variable u such that $y = tu$, which turns (4.27)-(4.28) into (1.4)-(1.5) with $k = 4$, $\beta = 2$ and $\alpha = 5$. By the previous theorem it has a positive solution denoted by u. We must prove that $y = tu(t)$ fulfills the boundary conditions. Since u is bounded in a neighborhood of 0 we have $\lim_{t\to 0} tu(t) = 0$. Moreover since $t^2u(t) \in H([1, \infty[)$ and this space is continuously embedded in $C([1, \infty[)$ we have, for $t \geq 1$

$$|t^2u(t)| \leq C$$

therefore,

$$\lim_{t\to\infty} |tu(t)| \leq \lim_{t\to\infty} C/t = 0.$$

■

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