Existence and L_∞ estimates for a class of singular ordinary differential equations

J. M. Gomes

Abstract

We prove the existence of a positive solution to an equation of the form $\frac{1}{\Phi(t)}(\Phi(t)u'(t))' = f(u(t))$ with Dirichlet conditions where the friction term Φ'/Φ is increasing. Our method combines variational and topological arguments and provides an L_∞ estimate of the solution.

Keywords: second order singular differential equation, variational methods, topological shooting.

2000 Mathematics Subject Classification. 34B18 34C11

1 Introduction and main result

We are interested in existence and estimation of the L_∞-norm of a positive solution to the problem

$$(\Phi(t)u'(t))' + \Phi(t)f(u(t)) = 0 \tag{1}$$

$$u(0) = u(1) = 0 \tag{2}$$

We suppose that $\Phi \in C^1(]0, 1])$,

$$\Phi \geq m > 0 \text{ in }]0, 1], \tag{3}$$

and

$$\frac{\Phi'(t)}{\Phi(t)} \text{ is an increasing function.} \tag{4}$$

Moreover we assume that f is locally Lipchitz,

$$f(0) = 0, \tag{5}$$

and that there exists $M_0 > 0$ such that

$$f(t) < 0 \text{ if } t < M_0, \text{ and } f(t) > 0 \text{ if } t > M_0. \tag{6}$$

Note that an equivalent formulation to problem (1)-(2) is

$$u''(t) + a(t)u'(t) + f(u(t)) = 0, \qquad u(0) = u(1) = 0$$

where the friction term $a(t)$ is an increasing function. There is a vast literature dealing with existence of solutions of singular boundary value problems (see for instance [1], [3], [4], [5] and the references therein). In this work, however, by restricting ourselves to a particular class of equations, we manage to provide L_∞ estimates to the solutions even when the non-linear term $f(.)$ has arbitrary growth. Our method is inspired by the topological shooting method (see for instance [2]) and the classical mountain pass theorem of Ambrosetti and Rabinowitz [6] and it is settled in Section 2. In section 3 we prove our main result:

Theorem 1 *Suppose that $\Phi \in C^1(]0,1])$, f is a locally Lipchitz continuous function and that conditions (3)-(6) are fulfilled. Moreover suppose that*

$$\frac{\Phi(s)}{\Phi(t)} \leq K \textit{ for some } K > 0 \textit{ and every } 0 < t \leq s \leq 1, \tag{7}$$

and that there exists a nonnegative $v \in H_0^1(]0,1[)$ such that

$$\frac{1}{2}\int_0^1 \Phi(t)v'(t)^2 dt < \int_0^1 \Phi(t)F(v(t))dt < \infty, \tag{8}$$

where $F(v) = \int_0^v f(s)ds$.

Then Problem (1)-(2) has a positive solution u such that $M_0 < \max u \leq \|v\|_\infty$.

As motivating examples we may consider $\Phi(t) = t^{-\alpha}$ or $\Phi(t) = \exp(t^{-\alpha})$ with $\alpha > 0$ (the reader may easily verify that these functions fulfill conditions (3), (4) and (7)).

I thank Prof. Luis Sanchez for helpful discussions and suggestions on this problem.

2 Variational setting and auxiliary results

Throughout this section we assume that there exist $m, m^*, L > 0$ such that, for all $t \in]0,1[$,

$$0 < m \leq \Phi(t) \leq m^* \ \ , \ \ |\Phi'(t)| \leq L, \tag{9}$$

and

$$\frac{\Phi'(t)}{\Phi(t)} \text{ is a strictly increasing function.} \tag{10}$$

Since we are looking for positive solutions we assume that f is extended by zero in $]-\infty, 0]$. The reader may easily verify that any non-trivial solution to (1)-(2) where f has been extended by zero should be positive in $]0,1[$ therefore being a solution of the initial problem. We shall consider the Sobolev space $H = H_0^1(]0,1[)$ consisting in absolutely continuous functions u such that

$$\|u\|^2 = \int_0^1 u'^2(t)dt \ < \infty \ , \ u(0) = u(1) = 0.$$

Problem (1)-(2) may be viewed as the Euler-Lagrange equation of the functional $J : H \to \mathbb{R}$ defined by:

$$J(u) = \frac{1}{2}\int_0^1 \Phi(t)u'^2(t)dt - \int_0^1 \Phi(t)F(u(t))dt$$

where $F(u) = \int_0^u f(s)ds$. We will suppose that J satisfies the following property:

$$\exists\, v \in H \ : J(v) < 0. \tag{11}$$

Remark 1 *Property (11) is trivially fulfilled if, for some $\epsilon > 0$, $f(u) \geq \epsilon u^\alpha - C$ for all $u \geq 0$, where $\alpha > 1$ and $C > 0$.*

Let $\overline{M} = \|v\|_\infty$. Since $J(w) \geq 0$ for every $w \in H$ with $\|w\|_\infty \leq M_0$, we have $\overline{M} > M_0$. Given $M \in [M_0, \overline{M}]$, we will consider the following subset of H:

$$\mathfrak{C}_M = \{u \in H : \max u \geq M\},$$

and the truncated functional $J_M : H \to \mathbb{R}$,

$$J_M(u) = \frac{1}{2}\int_0^1 \Phi(t)u'^2(t)dt - \int_0^1 \Phi(t)F_M(u(t))dt$$

where

$$F_M(u) = \begin{cases} F(u) \text{ if } u \leq M \\ F(M) \text{ if } u > M \end{cases}.$$

Remark 2 *From the compact injection of* $H_0^1(]0,1[)$ *in* $C([0,1])$ *we conclude that* $\mathfrak{C}_M$ *is weakly sequentially closed and that* J_M *is coercive and weakly lower semi-continuous.*

The main result of this section is the following proposition whose proof will become clear after a sequence of lemmas:

Proposition 2 *Let* $f \in C([0,+\infty[)$ *and* $\Phi \in C^1([0,1])$ *satisfying respectively properties (5)-(6) and (9)-(10). Moreover, suppose that* J *satisfies property (11) and let* $\overline{M} = \|v\|_\infty$*. Then there exists a classical positive solution* u *to problem (1)-(2) with* $M_0 < \max u \le \overline{M}$*.*

We will be interested in the family of minimizers u_M of J_M in $\mathfrak{C}_M$ where $M \in [M_0, \overline{M}]$. By remark 2 we know that u_M exists for every $M \in [M_0, \overline{M}]$. We also know that:

Lemma 3 *Let* u_M *be a minimizer of* J_M *in* $\mathfrak{C}_M$*. Then* $\max u = M$*.*

Proof. Given $w \in \mathfrak{C}_M$ define

$$\overline{w}(t) = \min\{w(t), M\}.$$

If $\overline{w} \neq w$ then,

$$\int_0^1 \Phi \overline{w}'^2 < \int_0^1 \Phi w'^2$$

and

$$\int_0^1 \Phi F_M(\overline{w}) = \int_0^1 \Phi F_M(w).$$

Therefore $J_M(\overline{w}) < J_M(w)$ and the lemma follows. ■

Given $M \in [M_0, \overline{M}]$, we define two types of minimizers of J_M in $\mathfrak{C}_M$:

Definition. Let u_M be a minimizer of J_M in $\mathfrak{C}_M$.

We say that u_M is a minimizer of type A if there exists a unique $t_0 \in]0,1[$ such that $u(t_0) = M$.

We say that u_M is a minimizer of type B if, given t_α(resp. t_β)= min(resp. max)$\{t : u_M(t) = M\}$, we have

$$u'_-(t_\alpha) = u'_+(t_\beta) = 0.$$

Remark 3 *If* u_M *is a minimizer of type A then* u *satisfies equation (1) in* $]0,t_0[\cup]t_0,1[$ *since* $J'_M(u)v = J'(u)v$ *for every* $v \in C_0^\infty(]0,t_0[\cup]t_0,1[)$*. For the same reason, if* v_M *is a type B minimizer, it satisfies equation (1) in* $]0,t_\alpha[\cup]t_\beta,1[$*. If* w *is simultaneously of type A and B, then* w *is a classical solution to problem (1)-(2).*

Lemma 4 *Let u be a minimizer of J_M in $\mathfrak{C}_M$. Then u is of type A or B (possibly both).*

Proof. We may rephrase the lemma as:

Let $t_\alpha\,(t_\beta) = \min\,(\max)\,\{t : u(t) = M\}$. Then either $u'_-(t_\alpha) = u'_+(t_\beta) = 0$ or $t_\alpha = t_\beta$.

Integrating equation (1) between t_1 and t_2 and letting $t_1, t_2 \to t_\alpha\ (t_\beta)$, we conclude that $u'_-(t_\alpha)\ (u'_+(t_\beta))$ is well defined. Suppose, in view of a contradiction, that $t_\alpha < t_\beta$ and $u'_-(t_\alpha) > 0$ (the case $u'_+(t_\beta) < 0$ is treated with similar arguments). Choose $\theta, \epsilon > 0$ such that $u'(t) \geq \theta$ for every $t \in]t_\alpha - \epsilon, t_\alpha[$ (we may suppose $\epsilon < t_\beta - t_\alpha$) and consider:

$$v_\epsilon(t) = -(|t - t_\alpha| - \epsilon)_-\,. \tag{12}$$

We assert that, for a small ϵ,

$$\lim_{s\to 0} \frac{J_M(u + sv_\epsilon) - J_M(u)}{s} < 0. \tag{13}$$

If (13) holds, then a for sufficiently small $s^* > 0$, we have $u + s^*v_\epsilon \in \mathfrak{C}_M$ (since $(u + s^*v_\epsilon)(t_\beta) = M$) and $J_M(u + s^*v_\epsilon) < J_M(u)$ which contradicts the assumption that u is a minimizer of J_M in $\mathfrak{C}_M$.

In fact, lemma 3 and (12) imply $u + s^*v_\epsilon \leq M$. Therefore

$$\lim_{s\to 0} \frac{J_M(u + sv_\epsilon) - J_M(u)}{s} = \lim_{s\to 0} \frac{J(u + sv_\epsilon) - J(u)}{s} =$$

$$= \int_0^1 \Phi(t)u'(t)v'_\epsilon(t)dt - \int_0^1 \Phi(t)f(u(t))v_\epsilon(t)dt \leq$$

$$-\theta \int_{t_\alpha-\epsilon}^{t_\alpha} \Phi(t) + \int_{t_\alpha}^{t_\alpha+\epsilon} \Phi(t)u'(t)dt - \int_{t_\alpha-\epsilon}^{t_\alpha+\epsilon} \Phi(t)f(u(t))v_\epsilon(t)dt.$$

We observe that, by (9),

$$-\theta \int_{t_\alpha-\epsilon}^{t_\alpha} \Phi(t) \leq -m\theta\epsilon \tag{14}$$

and for some $C > 0$ independent of ϵ,

$$\int_{t_\alpha-\epsilon}^{t_\alpha+\epsilon} \Phi(t)f(u(t))v_\epsilon(t)dt \leq C\epsilon^2. \tag{15}$$

Also, by Holder inequality, (9) and lemma 3,

$$\int_{t_\alpha}^{t_\alpha+\epsilon} \Phi(t)u'(t)dt = \int_{t_\alpha}^{t_\alpha+\epsilon} (\Phi(t_\alpha) + \Phi'(t_\alpha + \gamma(t)(t - t_\alpha))(t - t_\alpha))u'(t)dt \leq$$

$(0 < \gamma(t) < 1)$

$$\leq \Phi(t_\alpha)(u(t_\alpha + \epsilon) - M) + L\|u\|\epsilon^{\frac{3}{2}} \leq L\|u\|\epsilon^{\frac{3}{2}}. \tag{16}$$

Therefore, by (14), (15) and (16) we have

$$\lim_{s\to 0} \frac{J_M(u + sv_\epsilon) - J_M(u)}{s} \leq -m\theta\epsilon + C\epsilon^2 + L\|u\|\epsilon^{\frac{3}{2}},$$

and the assertion follows for sufficiently small ϵ. ■

In the next lemma we provide a sharper characterization of a type A minimizer.

Lemma 5 *Let u be a minimizer of J_M in $\mathfrak{C}_M$ of type A. Then*

$$(i)\, u'_-(t_0) > 0 \text{ and } u'_+(t_0) < 0 \quad \text{or} \quad (ii)\, u'(t_0) = 0.$$

Proof. In view of a contradiction, suppose that $u'_-(t_0) = 0$ and $u'_+(t_0) < 0$ (the reversed case is proved with similar arguments). Consider the following perturbation function:

$$w_\epsilon(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq t_0 - \epsilon \\ 4(t - t_0 + \epsilon) & \text{if } t_0 - \epsilon < t \leq t_0 - \epsilon/2 \\ -6(t - t_0 + \epsilon/2) + 2\epsilon & \text{if } t_0 - \epsilon/2 < t \leq t_0 \\ t - t_0 - \epsilon & \text{if } t_0 < t \leq t_0 + \epsilon \\ 0 & \text{if } t_0 + \epsilon < t \leq 1 \end{cases}.$$

Trivially, for sufficiently small ϵ, $w_\epsilon \in H$. Given $\lambda > 0$, since F_M is a Lipchitz function, we have for some $C_1 > 0$ independent of ϵ,

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \Phi(t)F_M(u + \lambda w_\epsilon)(t)dt \geq \int_{t_0-\epsilon}^{t_0+\epsilon} \Phi(t)F_M(u(t))dt - C_1\epsilon^2\lambda. \tag{17}$$

Also

$$\frac{1}{2}\int_{t_0-\epsilon}^{t_0+\epsilon} \Phi(t)(u' + \lambda w'_\epsilon)^2(t)dt =$$

$$\frac{1}{2}\int_{t_0-\epsilon}^{t_0+\epsilon} \Phi(t)u'^2(t)dt + \lambda\int_{t_0-\epsilon}^{t_0+\epsilon} \Phi(t)u'(t)w'_\epsilon(t)dt + \frac{\lambda^2}{2}\int_{t_0-\epsilon}^{t_0+\epsilon} \Phi(t)w'^2_\epsilon(t)dt \leq$$

$$\frac{1}{2}\int_{t_0-\epsilon}^{t_0+\epsilon} \Phi(t)u'^2(t)dt + \lambda\int_{t_0-\epsilon}^{t_0+\epsilon} \Phi(t)u'(t)w'_\epsilon(t)dt + C_2\lambda^2\epsilon, \tag{18}$$

where $C_2 = 36m^*$. Note that, by (17) and (18), we have

$$J_M(u + \lambda w_\epsilon) \leq J_M(u) + \Psi(\lambda, \epsilon) \tag{19}$$

with

$$\Psi(\lambda,\epsilon)=\lambda\int_{t_0-\epsilon}^{t_0+\epsilon}\Phi(t)u'(t)w'_\epsilon(t)dt+C(\lambda^2\epsilon+\epsilon^2\lambda),$$

where $C=\max\{C_1,C_2\}$. Our purpose is to show the existence of $\lambda,\epsilon>0$ such that $\Psi(\lambda,\epsilon)<0$ and $u+\lambda w_\epsilon\in\mathfrak{C}_M$, obtaining a contradiction from (19). Since $u'_-(t_0)=0$ and $u'_+(t_0)<0$, by (9) we may take $\epsilon_0,\theta>0$ such that, for every $0<\epsilon<\epsilon_0$,

$$|\Phi(s)u'(s){w_\epsilon}'(s)|<\theta/2 \text{ if } s\in[t_0-\epsilon,t_0]$$

and

$$\Phi(s)u'(s)\le-\theta \text{ if } s\in[t_0,t_0+\epsilon].$$

Then

$$\Psi(\lambda,\epsilon)\le-\frac{\theta}{2}\lambda\epsilon+C\lambda\epsilon(\epsilon+\lambda),$$

and, if we fix $\lambda=\frac{\theta}{4C}$, we have, for $\epsilon<\min\{\frac{\theta}{4C},\epsilon_0\}$,

$$\Psi(\lambda,\epsilon)<0. \tag{20}$$

In order to insure that $u+\lambda w_\epsilon\in\mathfrak{C}_M$, take ϵ_1 such that if $s\in]t_0-\epsilon_1,t_0[$, then

$$6\lambda-u'(s)>2\lambda. \tag{21}$$

We have, for $\epsilon<\min\{\frac{\theta}{4C},\epsilon_1,\epsilon_0\}$,

$$(u+\lambda w_\epsilon)(t_0-\frac{\epsilon}{2})=(u+\lambda w_\epsilon)(t_0)+\int_{t_0}^{t_0-\frac{\epsilon}{2}}(u'+\lambda w'_\epsilon)(s)ds=$$

$$M-\lambda\epsilon+\int_{t_0-\frac{\epsilon}{2}}^{t_0}(6\lambda-u'(s))ds,$$

and by (21),

$$(u+\lambda w_\epsilon)(t_0-\frac{\epsilon}{2})\ge M,$$

then $u+\lambda w_\epsilon\in\mathfrak{C}_M$ and the proof is concluded. ∎

In the next lemma we establish an important fact concerning the coexistence of type A and type B minimizers of J_M in $\mathfrak{C}_M$:

Lemma 6 *Suppose that for a certain $M\in[M_0,\overline{M}]$ there exist minimizers u and v of J_M in $\mathfrak{C}_M$ such that u is of type A and v is of type B. Then u is of type B (therefore being a classical solution to problem (1)-(2)).*

Proof. Since u is of type A let t_0 be the point where u equals M. Since v is of type B, let t_α(resp. t_β)= min(resp. max)$\{t : v(t) = M\}$ and

$$v'_-(t_\alpha) = v'_+(t_\beta) = 0.$$

We have $t_0 \leq t_\beta$ or $t_0 \geq t_\alpha$. Suppose, in view of a contradiction, that $t_0 \leq t_\beta$ and u is not of type B (the other case is proved with similar arguments). Then by Lemma 5 we have $u'_+(t_0) < 0$.

Claim: For every $t \in]t_\beta, 1[$, $u(t) < v(t)$. In particular, $u'(1) > v'(1)$.

Suppose that for some $t^* \in]t_\beta, 1[$ we had $u(t^*) = v(t^*)$ and $u'(t^*) > v'(t^*)$ (the case $u'(t^*) = v'(t^*)$ is excluded by the existence and uniqueness theorem). Moreover, suppose that

$$\frac{1}{2}\int_{t^*}^1 \Phi u'^2 - \int_{t^*}^1 \Phi F_M(u) \leq \frac{1}{2}\int_{t^*}^1 \Phi v'^2 dt - \int_{t^*}^1 \Phi F_M(v), \tag{22}$$

and let

$$v^*(t) = \begin{cases} v(t) \text{ if } & 0 \leq t \leq t^* \\ u(t) \text{ if } & t^* < t \leq 1 \end{cases}.$$

Then $v^* \in H$ and

$$J_M(v^*) \leq J_M(v),$$

therefore v^* is also a minimizer in $\mathfrak{C}_M$. This is absurd since v^* is not differentiable at t^* (see remark 3). In case where, instead of (22), we had the reversed inequality we get the same contradiction by considering:

$$u^*(t) = \begin{cases} u(t) \text{ if } & 0 \leq t \leq t^* \\ v(t) \text{ if } & t^* < t \leq 1 \end{cases}.$$

The strict inequality $u'(1) > v'(1)$ is a consequence of the existence and uniqueness theorem and the claim is proved.

Let

$$\hat{t} = \sup\{t : \ t_0 \leq t \leq 1 : \ u'(s) \leq 0 \ \forall s \in [t_0, t]\}.$$

We assert that $\underline{u(\hat{t}) < M_0 \text{ and } u'(\hat{t}) > v'(1)}$. In fact, if $\hat{t} = 1$ the assertion simply follows from the previous claim. If $\hat{t} < 1$, we may conclude from equation (1) and our assumptions on f that $u(\hat{t}) < M_0$ and $u'(\hat{t}) = 0$ (in fact, $u(\hat{t})$ is a local minimum of u).

Similarly, if we define

$$\bar{t} = \inf\{t : \ t_\beta \leq t \leq 1 \ v'(s) \leq 0 \ \forall s \in [t, 1]\},$$

we have that $\underline{v'(\bar{t}) = 0 \text{ and } v(\bar{t}) > M_0}$ (in fact, $v(\hat{t})$ is a local maximum of v). Then, if we consider in the phase plane (x, x') the curves U and V

corresponding to $u|_{[t_0,\hat{t}]}$ and $v|_{[\bar{t},1]}$, they must intersect at some point $P = (\mu,\mu')$ in the fourth quadrant. That is, for some $t_1 < t_2$,

$$u(t_1) = v(t_2) = \mu \text{ and } u'(t_1) = u'(t_2) = \mu' < 0.$$

Moreover we may suppose that P is such that μ is minimal. Let $T = t_2 - t_1$ and consider $v_T(t) = v(t+T)$. This translate of v satisfies, for every $t \in]t_1, 1-t_2+t_1[$,

$$(\Phi(t+T){v_T}'(t))' = -\Phi(t+T)f(v_T(t)),$$

or equivalently

$${v_T}'' = -f(v_T(t)) - {v_T}'(t)\frac{\Phi'(t+T)}{\Phi(t+T)},$$

with initial conditions $v_T(t_1) = u(t_1)$ and $v'_T(t_1) = u'(t_1)$. However, u is a solution to

$$u'' = -f(u(t)) - u'(t)\frac{\Phi'(t)}{\Phi(t)}.$$

Since $\frac{\Phi'}{\Phi}$ is strictly increasing and $-u'(t_1) = -v'_T(t_1) > 0$, we obtain

$$u''(t_1) < v''_T(t_1). \tag{23}$$

Again, by considering in the phase plane (x,x') the curves corresponding to $v_T|_{[t_1,1-t_2+t_1]}$ and $u|_{[t_1,1]}$, we conclude from (23) the existence of $(\hat{\mu},\hat{\mu}')$ in the fourth quadrant, with $\hat{\mu} < \mu$, such that, for some $t_1 < t' < 1-t_1+t_2$,

$$u(t_1) = v_T(t') = \hat{\mu} \text{ and } u'(t_1) = v_T(t') = \hat{\mu}',$$

or, for $t_3 = t' + t_2 - t_1$,

$$u(t_1) = v(t_3) = \hat{\mu} \text{ and } u'(t_1) = v'(t_3) = \hat{\mu}' < 0.$$

But this contradicts our assumption that μ is minimal. We may conclude that if $t_0 \le t_\beta$, u must be of type B.

If $t_0 \ge t_\alpha$ the proof is identical and we will just sketch it. By lemma 5 we have $u'(t_0) > 0$. With a similar reasoning to the one in the claim we may prove that $v'(0) > u'(0) > 0$. We define

$$\hat{t}^* = \inf\{t : \ 0 \le t \le t_0 \ , \ u'(s) \ge 0 \ \forall s \in [t,t_0]\}$$

and

$$\bar{t}^* = \sup\{t : \ 0 \le t \le t_\alpha \ , \ v'(s) \ge 0 \ \forall s \in [0,t]\}.$$

Then $u(\hat{t}^*) < M_0$ and $u'(\hat{t}^*) < v'(0)$. Also $v(\bar{t}^*) > M_0$ and $v'(\bar{t}^*) = 0$. Then the curves $u|_{[\hat{t}^*,t_0]}$ and $v|_{[0,\bar{t}^*]}$ must intersect at some point $P = (\nu, \nu')$ in the first quadrant of the phase plane (x, x'). Let us assume that ν is minimal and consider v_{-T} the translate of the left branch of v that, at some point t_1, coincides with u with same image and same positive derivative. Our assumption on the term $\frac{\Phi'}{\Phi}$ implies that $u''(t_1) < v''_{-T}(t_1)$ and we get the same contradiction. ∎

We are now in a position to prove proposition 2.

Proof of Proposition 2: Let $I = [M_0, \overline{M}]$ and consider the following subsets I_A and I_B:

$$I_A\,(I_B) = \{M \in [M_0, \overline{M}] : J_M \text{ has a minimizer in } \mathfrak{C}_M \text{ of type A (B)}\}.$$

By lemma 4 we have $I = I_A \cup I_B$. We assert that I_A and I_B are non-empty. We have $M_0 \in I_A$. In fact, let u_{M_0} be a minimizer of J_{M_0} in $\mathfrak{C}_{M_0}$. If u_{M_0} is of type B then by the existence and uniqueness theorem, $u_{M_0} \equiv M_0$, an obvious contradiction. Now, in order to prove that I_B is non-empty, we have $\overline{M} \in I_B$ or $\overline{M} \notin I_B$. Suppose that $\overline{M} \notin I_B$ and let $u_{\overline{M}}$ be a minimizer of $J_{\overline{M}}$ in $\mathfrak{C}_{\overline{M}}$. By lemma 5 there exists t_0 such that $u_{\overline{M}}(t_0) = \overline{M}$, $u'_{\overline{M}}(t_0)_- > 0$ and $u'_{\overline{M}}(t_0)_+ < 0$. Then, for sufficiently small $\lambda, \epsilon > 0$ we have

$$J_{\overline{M}}(u_{\overline{M}} + \lambda v_\epsilon) < J_{\overline{M}}(u_{\overline{M}}),$$

where v_ϵ was defined in (12) (see lemma 4 for details). Then, by (11),

$$\min J_{\overline{M}} < \min J_{\overline{M}}|_{\mathfrak{C}_{\overline{M}}} < 0,$$

and the continuous embedding of H in $C([0,1])$ implies that a minimum of $J_{\overline{M}}$ is a local minimum of J. It is therefore a nontrivial classical solution to (1)-(2) with $M_0 < \max u \leq \overline{M}$. In particular, it implies that I_B is nonempty.

Finally, we state that I_A and I_B are closed subsets of I. We will only consider I_A since the other case is identical. Let (M_n) be a sequence in I_A such that $M_n \to M$. Let u_n be a corresponding sequence of type A minimizers of J_{M_n} in $\mathfrak{C}_{M_n}$. Since (u_n) is trivially bounded we may extract a weakly convergent subsequence (still denoted by u_n) such that $u_n \rightharpoonup u$. Since the weak convergence implies L_∞ convergence one gets that $u \in \mathfrak{C}_M$ and u is of type A. It remains to show that u is a minimizer of J_M in $\mathfrak{C}_M$. Since

$$\lim_{n\to\infty} \int_0^1 \Phi F_{M_n}(u_n) = \int_0^1 \Phi F_M(u)$$

and

$$\int_0^1 \Phi u'^2 \leq \liminf_{n\to\infty} \int_0^1 \Phi {u_n}'^2,$$

we have

$$J_M(u) \leq \liminf J_{M_n}(u_n).$$

Moreover, if we consider the sequence $w_n = (M_n/M)u$, we have $w_n \to u$ in H and $w_n \in \mathfrak{C}_{M_n}$, for all $n \in \mathbb{N}$. Then

$$J_M(u) = \lim_{n\to\infty} J_{M_n}(w_n)$$

and

$$J_{M_n}(w_n) \geq J_{M_n}(u_n),$$

for all $n \in \mathbb{N}$. Consequently,

$$J_M(u) \geq \limsup_{n\to\infty} J_{M_n}(u_n) \geq \liminf_{n\to\infty} J_{M_n}(u_n) \geq J_M(u),$$

or

$$\lim_{n\to\infty} J_{M_n}(u_n) = J_M(u).$$

If, for a certain u^* in $\mathfrak{C}_M$, $J_M(u^*) < J_M(u)$, then, for sufficiently large n, we would have

$$J_{M_n}(w_n^*) < J_{M_n}(u_n),$$

where $w_n^* = (M_n/M)u^*$, a contradiction. Then $M \in I_A$ and I_A is closed. With a similar reasoning one proves that I_B is closed. Then, by connectedness we have $I_A \cap I_B \neq \varnothing$ and by lemma 6 we conclude the existence of a classical solution u with $\max u \in I_A \cap I_B$. ■

Remark 4 *Note that the existence of a solution to problem (1)-(2) under the assumptions of proposition 2 could have been proved with the use of the Mountain Pass theorem of Ambrosetti and Rabinowitz. However this theorem does not provide sharp estimates on the L_∞ norm of the solutions.*

3 Proof of the main result

In this section we extend proposition 2 to the case where Φ may have a singularity at zero. Our technique relies in a simple approximation procedure to problem (1)-(2). The results established in the previous section obviously remain true when initial conditions imposed to equation (1) are $u(a) = u(b) = 0$ (where $a < b$).

Proof of Theorem 1: We may assume that the function v given by (8) has support contained in $[\epsilon_0, 1]$, where $\epsilon_0 > 0$ is sufficiently small (the more general assertion can be obtained as a limit case). Take $n_0 \in \mathbb{N}$ such that $0 < \frac{1}{n_0} < \epsilon_0$, and for every $n > n_0$, define:

$$\Phi_n(t) = \exp(\frac{t^2}{2n})\Phi(t).$$

Note that $\frac{\Phi_n'}{\Phi_n}$ is strictly increasing and $\Phi_n \to \Phi$ uniformly. Then, taking $H = H_0^1(]\frac{1}{n}, 1[)$, (8) implies that (11) is fulfilled for large n if we consider the restriction of v to $[\frac{1}{n}, 1]$. We can therefore apply proposition 2 to the family of problems

$$(\Phi_n(t)u'(t))' + \Phi_n(t)f(u(t)) = 0$$
$$u(\frac{1}{n}) = u(1) = 0$$

obtaining a sequence of solutions (u_n) such that $M_0 < \max u_n \leq \overline{M}$. We may suppose that the u_n's are extended by zero in]0,1].

Claim: The sequence (u_n) is equicontinuous.

Let $t_n \in]\frac{1}{n}, 1[$ be such that $u_n(t_n) = \max u_{n[\frac{1}{n},1]}$. For any $t \in [\frac{1}{n}, 1]$, we have

$$\Phi_n(t)u_n'(t) = \int_t^{t_n} \Phi_n(s)f(u_n(s))ds.$$

If $\frac{1}{n} \leq t \leq t_n$, assumption (7) implies

$$|u_n'(t)| \leq \int_t^{t_n} \frac{\Phi_n(s)}{\Phi_n(t)}|f(u_n(s))|ds \leq \exp(\frac{1}{2n})K\overline{f}. \quad (24)$$

where $\overline{f} = \max_{[0,\overline{M}]} |f|$. Since $u_n(t_n) \geq M_0$ we may conclude the existence of $l > 0$ independent of n such that $t_n \geq l$. Then, for any $t_n \leq t \leq 1$, we have, by (3),

$$|u_n'(t)| \leq \int_{t_n}^1 \frac{\Phi_n(s)}{\Phi_n(t)}|f(u_n(s))|ds \leq (1-l)\exp(\frac{1}{2n})\frac{\overline{f}}{m}\max_{[l,1]}\Phi, \quad (25)$$

and the claim follows from (24)-(25).

We can therefore take a subsequence (still denoted by (u_n)) such that $u_n \to u$ uniformly and by standard arguments, u is a solution of (1)-(2). Since $M_0 < \max u_n \leq \overline{M}$ for all n, we conclude that u is nontrivial and $M_0 < \max u \leq \overline{M}$ (the case $\max u = M_0$ is excluded by the existence and uniqueness theorem). ∎

Remark 5 *If we consider the change of variables* $t = 1 - t'$ *we may restate theorem 1 for a class of functions* $\Phi \in C^1([0,1[)$ *having a singularity at 1:*

Suppose that f *satisfies (5)-(6) and:*

1)$\Phi(t) \geq m > 0 \ \forall t \in [0,1[$,

2)Φ'/Φ *is a decreasing function,*

3)$\frac{\Phi(t)}{\Phi(s)} \leq K$ *for some* $K > 0$ *and every* $0 \leq t \leq s < 1$,

4)There exists a nonnegative $v \in H_0^1(]0,1[)$ *such that*

$$\frac{1}{2}\int_0^1 \Phi(t)v'(t)^2 dt < \int_0^1 \Phi(t)F(v(t))dt < \infty,$$

where $F(v) = \int_0^v f(s)ds$.

Then Problem (1)-(2) has a positive solution u *such that* $M_0 < \max u \leq \|v\|_\infty$.

Remark 6 *Note that the method only requires that* $f > 0$ *in* $[M_0, \|v\|_\infty[$ *where* $\|v\|_\infty$ *is given by (8). The behaviour of* f *in* $[\|v\|_\infty, \infty[$ *is not relevant for our existence-estimation result.*

Acknowledgements
The author is supported by Fundação para a Ciência e a Tecnologia, [POCTI-FEDER].

References

[1] Positive solutions of Differential, Difference and Integral equations, R. Agarwal, D. O´Regan, P.Wong, Kluwer Academic Publishers (1999)

[2] H. Berestycki, P. L. Lions, L. A. Peletier, An ODE approach to the existence of positive solutions for semilinear problems in $\mathbb{R}^N$, Indiana Univ. Math. J.30 (1981), 141-157.

[3] Bobisud, L. E.; O'Regan, D. Positive solutions for a class of nonlinear singular boundary value problems at resonance. J. Math. Anal. Appl. 184 (1994), no. 2, 263–284.

[4] O'Regan, Solvability of some two point boundary value problems of Dirichlet, Neumann, or periodic type, Dynamical System Appl 2 (1993) 163-182

[5] O'Regan, Nonresonance and existence for singular boundary value problems, Nonlinear Anal. 23 (1994) ,nº2, 165-186

[6] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, Reg. Conf. Series in Math. 65, Amer. Math. Soc. 1986.

José Maria Gomes
CMAF-Faculdade de Ciências da Universidade de Lisboa
Avenida Professor Gama Pinto, 2, 1649-003 Lisboa
E-mail address: zemaria@cii.fc.ul.pt