ON THE ASYMPTOTIC BEHAVIOR OF THE SECOND MOMENT OF THE FOURIER TRANSFORM OF A RANDOM MEASURE

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Abstract. The behavior at infinity of the Fourier transform of the random measures that appear in the theory of multiplicative chaos of Mandelbrot, Peyrière and Kahane, is an area quite unexplored. For context and further reference, we present first an overview of this theory and then the result which is the main objective of this work generalizing a result previously announced by J.-P. Kahane. We establish an estimate for the asymptotic behavior of the second moment of the Fourier transform of the limit random measure in the theory of multiplicative chaos. In the last section, after looking at the behavior at infinity of the Fourier transform of some remarkable functions and measures, we prove a formula essentially due to Frostman involving the Riesz kernels and finally, we present a methodological remark on the connection between uniform continuity and behavior at infinity for an integrable function.

1. Introduction

The problem considered in the second section of this work admits a general formulation that can be stated as follows. A random measure is defined in the sense of a random object (see [9, p. 9]) by the action of a random operator on a usual Borel measure in a way such that its Fourier transform is almost surely a uniformly continuous and bounded function. A natural conjecture to be made is that the almost sure behavior, at infinity, of the Fourier transform of the random measure is somehow related to the behavior at infinity of the Fourier transform of the Borel measure used to build this random measure. A technique that has given good results in problems such as the one here presented goes as follows (see [10, p. 253–255, 265–267]). One gets first good estimates on the behavior of the moments of the random functions and then, by an accumulation argument, the almost sure behavior is obtained. The study of the asymptotic behavior of the second moment, besides the instrumental usefulness for the technique described, can give an idea of what to expect on the almost sure behavior.

1.1. **Multiplicative chaos: an overview.** For future reference and for an understandable context to the following let us explain briefly some of the most important ideas of the beautiful theory of multiplicative chaos. The main references of the plainly developed theory are the masterful expositions [9], [11] and [12]. The foundation stones of this circle of ideas may be traced to Mandelbrot's work of 1972 [23] criticizing ideas of Kolmogorov's model for turbulence (1962) and proposing a substitute framework by means of a limit lognormal model. In it, instead of having the average energy dissipation over a cube of fixed radius being lognormal, as in Kolmogorov's model, it is an approximate dissipation of energy depending on a continuous parameter that would be lognormal. In that way, and

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as explained by Mandelbrot in 1977 (see [24]) the normalized exponentiation of Gaussian processes could be a more adjusted interpretation of limit lognormal processes which would in turn be the correct version of the lognormal hypothesis of Kolmogorov. Later on in 1974, a couple of notes of Mandelbrot ([21],[22]) were followed by works of Kahane ([8]) and Peyrière ([27], [15]) developing some of the characteristic features of the theory. Extensions, refinements and a refutal of a conjecture by Kahane were provided later by other authors (see [2],[33], [3] and [38]).

For a basic start let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and consider $(X_n(t))_{n\in\mathbb{N}}$ a sequence of independent Gaussian centered random functions defined over T a locally compact metric space which for us will be the usual Euclidean normed space $\mathbb{R}^\nu$. We may then define the associated lognormal weights

$$P_n(t) := \exp\left(X_n(t) - \frac{1}{2}\mathbb{E}[X_n^2(t)]\right) .$$

Observing that $\mathbb{E}[P_n(t,\cdot)] = 1$ if we define

$$Q_n(t) := P_1(t) \cdot P_2(t) \cdots P_n(t) ,$$

then $(Q_n(t,\omega)_{n\in\mathbb{N},t\in\mathbb{R}^\nu,\omega\in\Omega}$ is a **positive** $\mathbb{R}^\nu$**-martingale**. That means that:

1. for each $t_0 \in \mathbb{R}^\nu$ fixed $(Q_n(t_0,\cdot)_{n\in\mathbb{N}}$ is a $\mathcal{C}$ martingale where the filtration $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ is naturally given by:
$$C_n = \sigma(\{X_m : m \leq n\}) ;$$
2. for almost all $\omega_0 \in \Omega$ we have that $(Q_n(\cdot,\omega_0))_{n\in\mathbb{N}}$ is a sequence of positive Borel functions on $\mathbb{R}^\nu$.

Take now σ a positive Radon measure on $\mathbb{R}^\nu$ (see [25, p. 9] or [20, p. 75]) and consider the sequence of random measures defined by $(Q_n\sigma)_{n\in\mathbb{N}}$. The following result ensures the existence of the weak limit of this sequence.

Theorem 1.1. [12, p. 12] *Under the condition*

$$r(t) := \mathbb{E}[Q_n(t,\cdot)] \in L^1(\sigma)$$

we have that $(Q_n\sigma)_{n\in\mathbb{N}}$ *converges weakly (that is, over* C_0 *the continuous functions on* $\mathbb{R}^\nu$ *having zero as a limit at infinity) almost surely to a random measure we designate by* S_σ*.*

As a consequence of this result we may define an operator Q on the positive Radon measures on $\mathbb{R}^\nu$, $M^+ = M^+(\mathbb{R}^\nu)$, into the space of random measures by

$$\forall\sigma \in M^+ \;\; Q\sigma = S_\sigma = \lim_{n\to+\infty} Q_n\sigma .$$

This is by definition the **multiplicative chaos** operator associated with $(X_n)_{n\in\mathbb{N}}$. A basic fundamental fact is that the distribution of the operator Q, namely the joint distribution of $(Q\sigma_1(B_1), Q\sigma_2(B_2), \ldots, Q\sigma_n(B_n))$ for all choices of n, $\sigma_1, \ldots, \sigma_n$, $B_1, \ldots, B_n$, depends only on

$$q(s,t) := \sum_{n=1}^{+\infty} p_n(t,s) \leq +\infty$$

where we suppose that $p_n(t,s) := \mathbb{E}[X_n(t) \cdot X_n(s)] \geq 0$.

In this work we will be particularly interested in the case where for a certain parameter $u > 0$ we have

$$q(s,t) = u \ln^+\left(\frac{1}{\|t-s\|_{\mathbb{R}^\nu}}\right) + \bigcirc(1) \tag{1.1}$$

which is a natural model for isotropic turbulence. Suppose that $r \in L^1(\sigma)$. In general there are two extreme cases concerning the image of operator Q.

1. Either $Q\sigma \equiv 0$ a.s. in which case we say that Q is **degenerate** on σ;
2. Or the martingale $(Q_n\sigma(B))_{n\in\mathbb{N}}$ converges in $L^1(\Omega)$ for each given Borel set B. This means that

$$\mathbb{E}[Q\sigma](B) = r\sigma B = \int_B r(t)d\sigma(t) ,$$

which we represent by

$$\mathbb{E}Q\sigma = \mathbb{E}[\int dS_\sigma] = \int r d\sigma \tag{1.2}$$

and is usually described by saying that Q is **fully acting** on σ or Q **lives on** σ. It is possible to show [12, p. 13] that for each $\varphi \in C_0$ we have

$$\mathbb{E}[\int \varphi dS_\sigma] = \int \varphi r d\sigma .$$

In the case where $q(s,t)$ is given by formula (1.1) every compact having Hausdorff dimension greater than $u/2$ supports measures such that $Q\sigma \neq 0$.

When dealing with moments it is particularly useful to consider the L^2 theory. One may say that Q is **strongly non degenerate** on σ if for every compact K of $\mathbb{R}^\nu$ we have

$$\mathbb{E}Q\sigma(K) = \sigma(K) .$$

The L^2 theory gives some conditions for Q to be strongly non degenerate on σ.

Theorem 1.2. [9, p. 133] *The following are equivalent.*

(i) *Q is strongly non degenerate in σ and moreover $\mathbb{E}[Q\sigma(K)]^2 < +\infty$;*

(ii)

$$\int_K \int_K e^{p_n(t,s)} d\sigma(t)d\sigma(s) = \bigcirc(1) ;$$

(iii)

$$\int_K \int_K e^{q(t,s)} d\sigma(t)d\sigma(s) < +\infty .$$

Under condition (iii) of theorem (1.2) if $k(t)$ is a complex bounded function over $\mathbb{R}^\nu$ we have

$$\mathbb{E}\left[\left|\int_K k(t)dS_\sigma\right|^2\right] = \int_K \int_K k(t)\overline{k(s)}e^{q(t,s)} d\sigma(t)d\sigma(s) . \tag{1.3}$$

For $q(s,t)$ given by formula (1.1) condition (iii) of theorem 1.2 says that σ has finite u-energy (see for a definition [25, p. 109] or theorem 3.1). As a consequence (see [12, p. 45]), for $u < d$ we have Q lives on σ whenever σ has finite u-energy.

Let us define the Fourier transform of the random measure S_σ. Under the main hypothesis of theorem 1.1, namely $r \in L^1(\sigma)$, we have almost surely

$$\lim_{n\to+\infty} \int Q_n d\sigma = S_\sigma(1) < +\infty ,$$

as a consequence of formula (1.2). We can then conclude that there is convergence over the bounded continuous functions on $\mathbb{R}^\nu$ ([20, p. 98]). As a consequence, the definition of the Fourier transform of the random measure S_σ is straightforward.

Definition 1. The **Fourier transform** $\hat{S}_\sigma$, of the random measure S_σ is by definition the map defined almost surely by:

$$\forall \xi \in \mathbb{R}^\nu \;\; \hat{S}_\sigma(\xi) = \lim_{n\to+\infty} \int_{\mathbb{R}^\nu} \exp(2\pi i \xi t) Q_n \sigma(dt) \; .$$

As usual (see [16, p. 132]), it is easily verified that almost surely $\hat{S}_\sigma$ is uniformly continuous and that for a bounded and positive Radon measure σ the map $\hat{S}_\sigma$ is almost surely bounded.

Remark 1. With additional hypothesis it can be verified that S_σ is a random measure in the sense of a measurable map taking its values in a (measurable) space of measures. More precisely, if the operator Q is strongly non degenerate, then the martingale defined for all $\varphi \in C_0(\mathbb{R}^\nu)$ by $(\int \varphi Q_n d\sigma)_{n\in\mathbb{N}}$ is a L^2 martingale. In fact for some constants c, A

$$\forall n \in \mathbb{N} \;\; \mathbb{E}\left[\left|\int \varphi Q_n d\sigma\right|^2\right] \leq c \int\int \exp(q_n(t,s)) d\sigma(t) d\sigma(s) \leq A < +\infty \; .$$

Following [26], we can say that the sequence of random measures $(Q_n\sigma)_{n\in\mathbb{N}}$, converges in quadratic mean (see [26, p. 49]). This shows that S_σ is a random measure when considered as a map defined on a probability space and taking its values in the space of the Radon signed measures, which is a measurable space, when endowed with the Borel σ algebra associated with the topology of *vague* convergence. The random measures associated to the multiplicative chaos are in this way and under some restrictive hypothesis nontrivial examples of random signed measures in the sense of Kallemberg and Oliveira.

2. On the asymptotic behavior

Asymptotic behavior of the Fourier transform of a measure is a classical subject in Harmonic Analysis ([17, p. 218], [25, p. 168], [32, p. 347–351, 360–364]). This subject has received some attention recently. In part due to the relevance for applications of the L^2 energy norm, the behavior of second order moments is particularly interesting (see, for instance [36], [19], [5] and [39]).

The main result furnished by the established theory on the asymptotic behavior of the Fourier transform of the random measure S_σ is the following.

Theorem 2.1. [9, p. 135] *If $q(s,t)$ given by formula (1.1) is a bounded C^∞ function with compact support and if σ has compact support and a C^∞ density with respect to the Lebesgue measure, then:*

$$\mathbb{E}\left[\left|\hat{S}_\sigma(\xi)\right|^2\right] \asymp \|\xi\|^{u-\nu} \; .$$

The main purpose of this work is to prove the following extension of this result. It gives the asymptotic behavior of the second moment in the case where σ is a positive Radon measure with compact support admitting an L^2 density.

Theorem 2.2. *If $q(s,t)$ given by formula (1.1) and σ is a positive measure Radon measure with compact support on $\mathbb{R}^\nu$, such that:*

$$\int_{\mathbb{R}^\nu} |\hat{\sigma}(x)|^2 dx < +\infty \;\; , \;\; \forall \xi \in \mathbb{R}^\nu \int_{\mathbb{R}^\nu} \frac{|\hat{\sigma}(x)|^2}{\|x-\xi\|^{\nu-u}} dx < +\infty \; .$$

and such that, the operator Q is strongly non degenerate on σ. We have then for some constants c and d that:

$$\mathbb{E}[\left|\hat{S}_\sigma(\xi)\right|^2] \leq \frac{1}{\|\xi\|^{\nu-u}} \left(c + d\,\|\xi\|^\nu \sup_{\|x-\xi\|<\frac{\|\xi\|}{2}} |\widehat{\sigma}(x)|^2\right) .$$

Proof. We use the result in [9] which says that

$$\mathbb{E}[|\hat{S}(\xi)|^2] = \mathbb{E}[|\int \exp(2\pi i t\xi) dS(t)|^2] = \int_{(\mathbb{R}^\nu)^2} \exp(2\pi i(t-s)\xi) e^{q(t,s)} d\sigma(t)\, d\sigma(s) \tag{2.1}$$

as a consequence of formula (1.3). We will deal first with the special case where σ admits a C^∞ density f with compact support. Suppose then that $d\sigma(t) = f(t)dt$. By a trivial change of variables and by Fubini's theorem we get:

$$\int_{\mathbb{R}^{2\nu}} \frac{e^{-2\pi i(t-s)\xi}}{||t-s||^u} d\sigma(t)\, d\sigma(s) = \int_{\mathbb{R}^{2\nu}} \frac{e^{-2\pi i v\xi}}{||v||^u} f(v+s) f(s) dv\, ds =$$
$$\int_{\mathbb{R}^\nu} \frac{e^{-2\pi i v\xi}}{||v||^u} (f * \check{f})(s) dv .$$

The hypotheses on f imply that $(f * \check{f})$ is a C^∞ function with compact support strictly positive in a neighborhood of zero. As a consequence of proposition 3, we get for some constant c:

$$\mathbb{E}[|\hat{S}(\xi)|^2] \leq \frac{c}{||\xi||^{\min(u, \frac{\nu+1}{2})}} ,$$

which is a weaker result than the one announced for $u \geq (\nu-1)/2$. The general case needs another kind of approach. We apply theorem 3.1 to the last term in formula (2.1) to get for some constant d:

$$\mathbb{E}[|\hat{S}(\xi)|^2] \leq d \int_{\mathbb{R}^\nu} \frac{|\hat{\sigma}(x)|^2}{||x-\xi||^{\nu-u}} dx .$$

Denote by I the following integral:

$$I = \int_{\mathbb{R}^n} \frac{|\widehat{\sigma}(x)|^2}{||x-\xi||^a}\, dx . \tag{2.2}$$

In order to obtain the asymptotic behavior of this integral we consider a point ξ, fixed in $\mathbb{R}^n$ and the partition of the domain of integration given by:

$$\mathbb{R}^n = B(0, \frac{||\xi||}{2}) \cup B(\xi, \alpha) \cup \{x \in \mathbb{R}^n : ||x|| \geq \frac{||\xi||}{2} , ||x-\xi|| \geq \alpha\} , \tag{2.3}$$

where α is a parameter we will deal with, below. Let I_1 (respectively I_2, I_3) be the integral of the function $\frac{|\widehat{\sigma}(x)|^2}{||x-\xi||^a}$ over the set on the left (respectively on the middle, on the right) of the partition (2.3). Then, it is clear that:

$$I_1 \leq \frac{1}{||\xi||^a} \int_{\frac{||x||}{||\xi||}<\frac{1}{2}} \frac{|\widehat{\sigma}(x)|^2}{|1-\frac{||x||}{||\xi||}|^a}\, dx \leq \frac{1}{||\xi||^a}\, 2^a ||\widehat{\sigma}||_2^2 , \tag{2.4}$$

$$I_2 \leq \sup_{||x-\xi||<\alpha} |\widehat{\sigma}(x)|^2 \int_{||x||<\alpha} \frac{dx}{||x||^a} \leq \alpha^{\nu-a} \sup_{||x-\xi||<\alpha} |\widehat{\sigma}(x)|^2 \int_{||x||<\alpha} \frac{dx}{||x||^a} , \tag{2.5}$$

$$I_3 \leq \frac{1}{\alpha^a} \int_{||x||>\frac{||\xi||}{2}} |\widehat{\sigma}(x)|^2\, dx \leq \frac{||\widehat{\sigma}||_2^2}{\alpha^a} . \tag{2.6}$$

As a consequence, for some constants c and d and choosing $\alpha = \frac{||\xi||}{2}$ in (2.5) and in (2.6) we have that:

$$I \leq \frac{1}{||\xi||^a} \left(c + d\,||\xi||^\nu \sup_{||x-\xi||<\frac{||\xi||}{2}} |\widehat{\sigma}(x)|^2\right) , \tag{2.7}$$

as desired. □

Remark 2. Let σ be the Lebesgue measure concentrated on the unit ball of $\mathbb{R}^\nu$. As a consequence of formula (3.1) we will have that for some constant c:

$$|\hat{\sigma}(x)|^2 \leq \frac{c}{||x||^{\nu+1}} ,$$

And as a consequence:

$$\mathbb{E}[|\hat{S}(\xi)|^2] \leq \frac{c}{||\xi||^{\nu-u}} ,$$

in agreement with the result stated in ([9, p. 30]).

Remark 3. The final conclusion in the statement of theorem (2.2) clearly depends on the asymptotic behavior of the Fourier transform of the measure σ. We present next an example, (given by [29]), that shows that in general the integral in (2.2) has no rate of decay better than $\bigcirc(1)$. We will see that the measure under scrutiny hasn't compact support. As a consequence, a natural question is to find an example such as the one presented but with a measure with compact support.

Consider a sequence of functions $(\varphi_n)_{n\in\mathbb{N}}$ defined by:

$$\forall n \in \mathbb{N} \;\; \varphi_n = \mathbb{I}_{[-n,1-n]} + \mathbb{I}_{[n-1,n]} .$$

As φ_n is an even function its Fourier $\widehat{\varphi_n}$ transform is real valued. A quick computation shows that:

$$\forall n \in \mathbb{N} \;\; \widehat{\varphi_n}(\xi) = \frac{2\sin(\pi\xi)\cos((2n-1)\pi\xi)}{\pi\xi} .$$

Define now a sequence of functions $(\psi_n)_{n\in\mathbb{N}}$ by:

$$\forall n \in \mathbb{N} \;\; \psi_n = \varphi_n * \varphi_n .$$

A simple but tedious computation shows that $\psi_n(x)$ is a linear by pieces continuous function with compact support, simply described as the sum of three tent functions given by:

$$\begin{aligned}\psi_n(x) = {} & (2n+x)\mathbb{I}_{[-2n,-2n+1]}(x) + (-x-2n+2)\mathbb{I}_{[-2n+1,-2n+2]}(x) + \\ & (2x+2)\mathbb{I}_{[-1,0]}(x) + (2-2x)\mathbb{I}_{[0,1]}(x) + \\ & (x-2n+2)\mathbb{I}_{[2n-2,2n-1]}(x) + (2n-x)\mathbb{I}_{[2n-1,2n]}(x) .\end{aligned}$$

As $\widehat{\psi_n} = (\widehat{\varphi_n})^2$, we have that:

$$\forall n \in \mathbb{N} \;\; \widehat{\psi_n}(\xi) = \frac{4\sin^2(\pi\xi)\cos^2((2n-1)\pi\xi)}{\pi^2\xi^2}$$

This shows that ψ_n belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Take a sequence $(a_n)_{n\in\mathbb{N}^*}$ of non negative numbers such that $\sum_{n=1}^{+\infty} a_n < +\infty$ and define a measure $d\sigma(\xi) = f(\xi)d\xi$ with:

$$\forall \xi \in \mathbb{R} \;\; f(\xi) = \sum_{n=1}^{+\infty} a_n \widehat{\psi_n}(\xi) .$$

As $\sigma(\mathbb{R}) = 2\sum_{n=1}^{+\infty} a_n$, the measure σ is finite. Moreover, as a consequence of Cauchy-Schwarz inequality, the density of σ with respect to the Lebesgue measure is in $L^2(\mathbb{R})$. In fact, we have:

$$(\int_{\mathbb{R}} f^2(\xi)d\xi)^{\frac{1}{2}} = \left(\sum_{n,m=1}^{+\infty} a_n a_m \int_{\mathbb{R}} \widehat{\psi_n}(\xi)\widehat{\psi_m}(\xi)d\xi\right)^{\frac{1}{2}} \leq$$

$$||\psi_n||_2||\psi_m||_2(\sum_{n=1}^{+\infty} a_n^2)^{\frac{1}{2}}(\sum_{m=1}^{+\infty} a_m^2)^{\frac{1}{2}} < +\infty \ .$$

Observe now that for every $p \in \mathbb{N}$:

$$|\widehat{\sigma}(x)| = \frac{1}{2\pi}\sum_{n=1}^{+\infty} a_n\psi_n(x) \geq a_p\psi_p(x) \ .$$

As a consequence, for some constant c:

$$I(2n-1) = \int_{\mathbb{R}} \frac{|\widehat{\sigma}(x)|^2}{|x-(2n-1)|^a}dx \geq \int_{2n-2}^{2n} |\widehat{\sigma}(x)|^2 \geq ca_n^2$$

Finally, by choosing for example:

$$a_n = \begin{cases} \frac{1}{k^2}, & \text{if } n = 2^{2^k} \\ 0, & \text{otherwise} \end{cases} ,$$

we see that I has no rate of decay better than $\bigcirc(1)$.

3. Auxiliary results and methodological remarks

In this section we develop some results that were used in the proof of theorem 2.2 and that may be considered interesting in themselves.

3.1. **Some remarkable Fourier transforms.** For completeness's sake we next state and prove a classical result (see [4] or [35, p. 51]) on the asymptotic behavior of the Fourier transform of the indicator function of the unit ball in $\mathbb{R}^n$. The method used in the proof will be next applied to study a similar but less classical result. Hereafter in this chapter, letters c, d and e will denote constants not necessarily the same at every instance.

Proposition 1. *Let $B = B(0,1) = \{x \in \mathbb{R}^n \ : \ ||x|| < 1\}$ be the unit ball of $\mathbb{R}^n$ and $U = \mathbb{I}_B$ the indicator function of B. Then, for some constant c, we have that:*

$$|\widehat{U}(\xi)| \leq \frac{c}{|\xi|^{\frac{n+1}{2}}} \tag{3.1}$$

Proof. As U is a radial function its Fourier transform is given by (see [32, p. 430] and for a proof [31, p. 155]):

$$\widehat{U}(\xi) = 2\pi||\xi||^{\frac{2-n}{2}} \int_0^{+\infty} J_{\frac{n-2}{2}}(2\pi||\xi||r)\, U_0(r)\, r^{\frac{n}{2}}\, dr \ , \tag{3.2}$$

where U_0 is such that $U(x) = U_0(||x||)$ and, $J_{\frac{n-2}{2}}$ is a Bessel function. As U_0 is the indicator function of $[0,1[$ then:

$$\widehat{U}(\xi) = 2\pi||\xi||^{\frac{2-n}{2}} \int_0^1 J_{\frac{n-2}{2}}(2\pi||\xi||r)\, r^{\frac{n}{2}}\, dr \ .$$

By a change of variables given by $2\pi||\xi r|| = \rho$ this expression is turned on the following one:

$$\widehat{U}(\xi) = (2\pi)^{-\frac{n}{2}}||\xi||^{-n} \int_0^{2\pi||\xi||} J_{\frac{n-2}{2}}(\rho)\,\rho^{\frac{n}{2}}\,d\rho\ .$$

Now, a classical relation of Bessel functions (see [18, p. 141]) states that:

$$t^{\nu+1}\,J_\nu(t) = \frac{d}{dt}[\,t^{\nu+1}J_{\nu+1}(t)]\ . \tag{3.3}$$

Applying this relation with $\nu = \frac{n}{2} - 1$, gives:

$$\widehat{U}(\xi) = ||\xi||^{-\frac{n}{2}}\,J_{\frac{n}{2}}(2\pi||\xi||)$$

The asymptotic behavior of $J_\nu(t)$ is known (see [18, p. 134]). In fact, as for some constant c, the following estimate holds

$$|J_\nu(t)| \leq \frac{c}{\sqrt{t}} \tag{3.4}$$

we have finally that

$$|\widehat{U}(\xi)| \leq (\frac{c}{\sqrt{2\pi}})\,\frac{1}{||\xi||^{\frac{n+1}{2}}}\ ,$$

which is the result requested. □

A similar result is attained when the unit ball is replaced by $B(0,\delta)$ and the indicator function appears multiplied by a remarkable locally integrable radial function.

Proposition 2. *Let $B_\delta = B(0,\delta) = \{x \in \mathbb{R}^n : ||x|| < \delta\}$ be a ball of $\mathbb{R}^n$, centered in zero with radius $\delta > 0$ and, for $0 < \alpha < n$, the function defined by:*

$$U^\alpha_\delta(x) = \frac{\mathbb{I}_{B_\delta}(x)}{||x||^\alpha}\ .$$

Then, for some constants which we denote always by c we have that:

$$|\widehat{U^\alpha_\delta}(\xi)| \leq \begin{cases} \frac{c}{||\xi||^{n-\alpha}} & if \quad \alpha > \frac{n-1}{2} \\ \frac{c}{||\xi||^{\frac{n+1}{2}}} & if \quad \alpha \leq \frac{n-1}{2} \end{cases} \tag{3.5}$$

Proof. We will consider that $\delta = 1$. In fact, the change of variables given by $x = \delta y$ shows that:

$$\int_{||x||<\delta} \frac{e^{-2\pi i x\cdot\xi}}{||x||^\alpha}\,dx = \delta^{n-\alpha} \int_{||y||<1} \frac{e^{-2\pi i y\cdot(\delta\xi)}}{||y||^\alpha}\,dy\ .$$

As that means that

$$\widehat{U^\alpha_\delta}(\xi) = \delta^{n-\alpha}\,\widehat{U^\alpha_1}(\delta\xi)\ , \tag{3.6}$$

if for some constant c we have:

$$|\widehat{U^\alpha_1}(\xi)| \leq \frac{c}{||\xi||^{\min(n-\alpha,\frac{n+1}{2})}}\ ,$$

we will have also that

$$|\widehat{U^\alpha_\delta}(\xi)| \leq \frac{c\cdot\delta^{\max(0,\frac{n-1}{2}-\alpha)}}{||\xi||^{\min(n-\alpha,\frac{n+1}{2})}}\ .$$

We will use this remark later on. By using formula (3.2) again, and then a change of variables given by $2\pi||\xi||r = \rho$ in the integral obtained, the Fourier transform of $\widehat{U_1^\alpha}$ can be written in the following form:

$$\widehat{U_1^\alpha}(\xi) = \frac{(2\pi)^{\alpha-\frac{n}{2}}}{||\xi||^{n-\alpha}} \int_0^{2\pi||\xi||} J_{\frac{n-2}{2}}(\rho)\,\rho^{\frac{n}{2}-\alpha}\,d\rho\,. \tag{3.7}$$

We observe that by using the estimate given by (3.4) the integral on the right-hand side above converges absolutely at $+\infty$ if $\alpha > \frac{n+1}{2}$. Under that condition on α, the first line in formula (3.5) is obtained. In order to deal with a second instance of the first line of the result and with the second line of the result we have to prove, we notice that as a consequence of formula (3.3) and of a trivial integration by parts:

$$\int_0^z \rho^\mu\,J_\nu(\rho)\,d\rho = z^\mu J_{\nu+1}(z) - (\mu-\nu-1)\int_0^z \rho^{\mu-1}\,J_{\nu+1}(\rho)\,d\rho\,. \tag{3.8}$$

(See again [18, p. 141]). Suppose now that:

$$\frac{n-1}{2} < \alpha \leq \frac{n+1}{2}\,.$$

Applying formula (3.8) to the integral in formula (3.7) we get, for some constants c and d, that:

$$|\widehat{U_1^\alpha}(\xi)| \leq \frac{c}{||\xi||^{\frac{n+1}{2}}} + \frac{d}{||\xi||^{n-\alpha}} \int_0^{2\pi||\xi||} J_{\frac{n}{2}}(\rho)\,\rho^{\frac{n}{2}-\alpha-1}\,d\rho\,, \tag{3.9}$$

where the integral on the right-hand side of the expression converges absolutely by force of the condition on α. This condition obviously implies also that $n-\alpha < \frac{n+1}{2}$ and so, the first line of the statement of the proposition also holds in this case. Suppose now that:

$$\frac{n-3}{2} < \alpha \leq \frac{n-1}{2}\,.$$

We apply again formula (3.8) but this time to the integral in the right-hand side of the expression (3.9) and we get, for some constants c, d and e:

$$|\widehat{U_1^\alpha}(\xi)| \leq \frac{c}{||\xi||^{\frac{n+1}{2}}} + \frac{d}{||\xi||^{\frac{n+3}{2}}} + \frac{e}{||\xi||^{n-\alpha}} \int_0^{2\pi||\xi||} J_{\frac{n}{2}+1}(\rho)\,\rho^{\frac{n}{2}-\alpha-2}\,d\rho\,, \tag{3.10}$$

the integral, on the right-hand side, being absolutely convergent by the conditions imposed on α. These conditions also imply that $\frac{n+1}{2} \leq n-\alpha < \frac{n+3}{2}$ and so, in this case, we have as a conclusion the second line of the statement made in the proposition. If we suppose that the conditions on α are now given by:

$$\frac{n-5}{2} < \alpha \leq \frac{n-3}{2}\,,$$

the same method gives a new term in formula (3.10), namely, for some constants c, d, e, f:

$$|\widehat{U_1^\alpha}(\xi)| \leq \frac{c}{||\xi||^{\frac{n+1}{2}}} + \frac{d}{||\xi||^{\frac{n+3}{2}}} + \frac{e}{||\xi||^{\frac{n+5}{2}}} + \frac{f}{||\xi||^{n-\alpha}} \int_0^{2\pi||\xi||} J_{\frac{n}{2}+2}(\rho)\,\rho^{\frac{n}{2}-\alpha-3}\,d\rho\,, \tag{3.11}$$

where the integral is absolutely convergent and $\frac{n+3}{2} \leq n-\alpha < \frac{n+5}{2}$ by force of the conditions imposed on α. In order to conclude it is only necessary to proceed by induction observing that after a finite number of steps α will be close to its inferior limit, namely zero, and the integral appearing as a remainder is absolutely convergent. Observe that the integrals are all convergent at zero by virtue of the fact that for small ρ:

$$J_\nu(\rho) \approx \frac{\rho^\nu}{2^\nu\,\Gamma(1+\nu)}$$

see again [18, p. 134]. □

The following consequence of the last proposition was proven useful when dealing with a particular case of our ultimate goal, theorem 2.2.

Proposition 3. *Let g be a $C^\infty(\mathbb{R}^n)$ function with compact support such that g is strictly positive in a neighborhood of zero. Let $0 < \alpha < n$ and define $I_{\alpha,g}$, a Fourier transform, by:*

$$I_{\alpha,g}(\xi) = \int_{\mathbb{R}^n} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^\alpha}\, g(x)\, dx\ . \tag{3.12}$$

The asymptotic behavior of $I_{\alpha,g}$ is the same as the asymptotic behavior of $\widehat{U_1^\alpha}$, that is, for some constant c,

$$|I_{\alpha,g}(\xi)| \leq \frac{c}{||\xi||^{min(n-\alpha,\frac{n+1}{2})}}\ . \tag{3.13}$$

Proof. The conditions imposed on g, insure the existence of $M > 0$ and $0 < \eta < 2$ such that for $v \in B(0,\eta)$ we have that $g(0) > 0$,

$$g(v) = g(0) + dg(0)(v) + ||v||\epsilon(v)\ , \tag{3.14}$$

where $dg(0)$ is the differential of g at zero, $\epsilon(v)$ tends to zero with v and:

$$|dg(0)(v) + ||v||\epsilon(v)| \leq M\ . \tag{3.15}$$

Let $R > 0$ be such that the support of g is contained in $B(0,R)$. We can consider now a standard partition of unity subordinated to the open cover of the support of g given by $B(0,\eta)$ and $B(0,2R)\cup(B(0,\frac{\eta}{2}))^c$, (see [28, p. 162] for instance). More precisely, let ϕ_1,ϕ_2 be $C^\infty(\mathbb{R}^n)$ functions with compact support such that $\phi_1 \equiv 1$ in $B(0,\frac{\eta}{2})$ and $\text{supt}(\phi_1) \subset B(0,\eta)$, $\phi_2 \equiv 1$ in $B(0,R)$ and $\text{supt}(\phi_2) \subset B(0,2R)$. It is clear that if we define ψ_1,ψ_2 by $\phi_1 \equiv \psi_1$ and $\psi_2 \equiv (1-\phi_1)\cdot\phi_2$ then, ψ_1,ψ_2 are $C^\infty(\mathbb{R}^n)$ functions with compact support, verifying: $\psi_1 + \psi_2 \equiv 1$ on $B(0,R)$ and $\text{supt}(\psi_1+\psi_2) \subset B(0,2R)$.

The Fourier transform whose asymptotic behavior we pretend to study can now be written, using this partition of unity, as:

$$I_{\alpha,g}(\xi) = \int_{\mathbb{R}^n} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^\alpha}\, g(v)\cdot\psi_1(v)\, dv + \int_{\mathbb{R}^n} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^\alpha}\, g(v)\cdot\psi_2(v)\, dv\ . \tag{3.16}$$

Denote by $I^l_{\alpha,g}(\xi)$ (respectively $I^r_{\alpha,g}(\xi)$) the integral on the left (respectively on the right) on the right-hand side of equality (3.16). Observe that as $f(v) = \frac{g(v)\cdot\psi_2(v)}{||x||^\alpha}$ is a $C^\infty(\mathbb{R}^n)$ function with compact support its Fourier transform as given by $I^r_{\alpha,g}(\xi)$, has a decay at infinity as fast as we want as a consequence of, for instance, Paley-Wiener's theorem (see [28, p. 198]). There exists then a constant c such that:

$$|I^r_{\alpha,g}(\xi)| \leq \frac{c}{||\xi||^n}\ . \tag{3.17}$$

The integral $I^l_{\alpha,g}(\xi)$ can be further decomposed as follows:

$$I^l_{\alpha,g}(\xi) = \int_{||v||<\frac{\eta}{2}} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^\alpha}\, g(v)\, dv + \int_{\frac{\eta}{2}\leq||v||<\eta} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^\alpha}\, g(v)\cdot\psi_1(v)\, dv\ . \tag{3.18}$$

Using now (3.14), the integral on the left can be rewritten as:

$$\int_{||v||<\frac{\eta}{2}} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^{\alpha}} g(v)\,dv = g(0)\cdot \widehat{U^{\alpha}_{\frac{\eta}{2}}}(\xi) + \int_{||v||<\frac{\eta}{2}} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^{\alpha}} (dg(0)(v) + ||v||\epsilon(v))\,dv\,. \tag{3.19}$$

By passing to polar coordinates and on account of (3.15), we can estimate the integral on the right-hand side of this equality, by:

$$|\int_{||v||<\frac{\eta}{2}} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^{\alpha}}\,(dg(0)(v) + ||v||\epsilon(v))\,dv| \leq \frac{M}{2^{n-\alpha+1}\,(n-\alpha+1)}\,\eta^{n-\alpha+1}\,. \tag{3.20}$$

By passing to polar coordinates we can also take care of the integral on the right in (3.18). In fact, for some constant c:

$$|\int_{\frac{\eta}{2}\leq||v||<\eta} \frac{e^{-2\pi i\,\xi\cdot x}}{||x||^{\alpha}}\,g(v)\psi_2(v)\,dv| \leq c \sup_{||v||\leq\eta} |g(v)\psi_2(v)| \int_{\eta}^{\frac{\eta}{2}} \rho^{n-\alpha-1}\,d\rho \leq c\,\eta^{n-\alpha}\,. \tag{3.21}$$

In order to conclude, it is sufficient to collect the estimates made on (3.17), (3.19), (3.20) and (3.21), to recall proposition (2), in particular the remark at the beginning of the corresponding proof in conjunction with the fact that we took $0<\eta<2$, and we get for some constants c and d:

$$|I_{\alpha,g}(\xi)| \leq \frac{c}{||\xi||^{\min(n-\alpha,\frac{n+1}{2})}} + d\,\eta^{n-\alpha}\,. \tag{3.22}$$

Observing that in obtaining estimates (3.20) and (3.21) we only consider the modulus of the function we were integrating, we can take the parameter η as small as we want in (3.22), thus getting the result stated. □

3.2. **Some Parseval formulas and tempered distributions.** The following result was used in the proof of the main theorem to express the second moment of the Fourier transform of the random measure S_σ.

For $0<\alpha<n$ let U^α denote the locally integrable function, defined by:

$$U^{\alpha}(x) = \frac{1}{||x||^{\alpha}}\,\mathbb{I}_{\mathbb{R}^{*n}}(x)\,,$$

where $\mathbb{R}^{*n} = \mathbb{R}^n - \{0\}$. This function defines a tempered distribution whose Fourier transform denoted by $\widehat{U^\alpha}$ but also by $\mathcal{F}U^\alpha$ is represented again by a locally integrable function given by:

$$\widehat{U^{\alpha}}(\xi) = \frac{c(\alpha)}{||\xi||^{n-\alpha}}\,\mathbb{I}_{\mathbb{R}^{*n}}(\xi) \quad , \quad c(\alpha) = \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}-\alpha}\Gamma(\frac{\alpha}{2})}\,.$$

(See [30, p. 117] or [37, p. 52, 278]).

The following result essentially given by Frostman is usually formulated for real measures and with no exponential term in formula (3.24) (see [1, p. 22]).

Theorem 3.1. *: Let σ be a positive Radon measure over $\mathbb{R}^n$ with compact support and $0<\alpha<n$ such that E_α, the α energy of σ, is finite. That is:*

$$E_{\alpha} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{d\sigma(t)\,d\sigma(s)}{||t-s||^{\alpha}} < +\infty\,. \tag{3.23}$$

We then have:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{e^{2i\pi\xi(t-s)}}{||t-s||^{\alpha}} d\sigma(t)\,d\sigma(s) = c(\alpha)\int_{\mathbb{R}^n} \frac{|\widehat{\sigma}(x)|^2}{||x-\xi||^{n-\alpha}}dx\,, \tag{3.24}$$

whenever the integral on the right is finite.

Proof. The formula we have to prove is verified for measures given by $d\sigma(t) = \phi(t)dt$, where $\phi \in \mathcal{S}$, $\mathcal{S}$ being the Schwartz test function space of rapidly decreasing functions. Indeed, for such a measure the integral on the left-hand side of formula (3.24) which we denote by I is written as:

$$I = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{e^{2i\pi\xi t}}{||t-s||^\alpha} \phi(t)dt \right) e^{-2i\pi\xi s}\phi(s)ds = \int_{\mathbb{R}^n} \frac{(2\pi)^{n-\alpha}}{c(n-\alpha)} I_{n-\alpha}(h)(s)\overline{h}(s)ds \ ,$$

where, for $0 < \beta < n$, the β Riesz potential of f is given by:

$$I_\beta(f)(x) = \frac{c(\beta)}{(2\pi)^\beta} \int_{\mathbb{R}^n} \frac{f(y)}{||x-y||^{n-\beta}} \, dy$$

and h stands for $h(s) = e^{2i\pi\xi s}\phi(s)$. Now, given $f, g \in \mathcal{S}$ we have that:

$$\int_{\mathbb{R}^n} I_\beta(f)(x)\overline{g}(x)dx = \int_{\mathbb{R}^n} \frac{\widehat{f}(x)\widehat{\overline{g}}(x)}{(2\pi||x||)^\beta} dx \ ,$$

which is essentially Parseval's formula (see [30, p. 117] for all the properties of the notion of Riesz potential used). Observing that $\widehat{h}(y) = \widehat{\phi}(y-\xi)$ and $\widehat{\overline{h}} = \widehat{\phi}(\xi - y)$, we have:

$$I = \frac{1}{c(n-\alpha)} \int_{\mathbb{R}^n} \frac{\widehat{\phi}(y-\xi)\widehat{\phi}(\xi-y)}{||y||^{n-\alpha}} dy \, ,$$

which gives the result claimed in the statement of the theorem by a trivial change of variables, noticing that, as ϕ is real:

$$\widehat{\phi}(u)\widehat{\phi}(-u) = \widehat{\phi}(u)\overline{\widehat{\phi}}(u) = |\widehat{\phi}(u)|^2 \ ,$$

and that $C(\alpha) = 1/c(n-\alpha)$. Let now μ, be a complex measure with compact support and, for $0 < \beta < n$, define I_β, the β Riesz potential of μ, by:

$$I_\beta(\mu)(x) = \frac{c(\beta)}{(2\pi)^\beta} \left(\mu * U^{n-\alpha} \right) .$$

This definition makes good sense as we are considering the convolution of μ, a distribution with compact support, with $U^{n-\alpha}$ which is a tempered distribution. Observe that as a consequence of a theorem of Sobolev (see [34, p. 181]), $I_\beta(\mu)$ is locally in L^q for $q < \frac{n}{n-\beta}$ and in particular $I_\beta(\mu)$ is integrable over any compact of $\mathbb{R}^n$. Moreover, the Fourier transform of $I_\beta(\mu)$ in the sense of distributions is easily computed (see for instance [7, p. 21]) to give:

$$\widehat{I_\beta(\mu)} = \frac{c(\beta)}{(2\pi)^\beta} \widehat{\mu} \cdot \widehat{U^{n-\beta}} = \frac{1}{(\pi)^\beta} \widehat{\mu} \cdot \widehat{U^{n-\beta}} \ .$$

Considering now $d\mu(t) = e^{2\pi i \xi \cdot t} d\sigma(t)$ we have that:

$$I = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{d\mu(t)}{||t-s||^\alpha} \right) d\overline{\mu}(s) = \frac{(2\pi)^\alpha}{c(n-\alpha)} \int_{\mathbb{R}^n} I_{n-\alpha}(\mu)(s) d\overline{\mu}(s) \ .$$

As μ has compact support and as a consequence of the hypotheses done on the integral on the right of formula (3.24) we can apply Parseval formula (as given for instance in [16, p. 132] or [20, p. 121]) to obtain

$$I = \frac{(2\pi)^\alpha}{c(n-\alpha)} \int_{\mathbb{R}^n} \widehat{I_{n-\alpha}(\mu)}(x)\widehat{\overline{\mu}}(x)dx = c(\alpha) \int_{\mathbb{R}^n} \widehat{\mu}(x) U^\alpha(x) \widehat{\overline{\mu}}(-x) dx \ ,$$

which, after some computations of Fourier transforms and a change of variables, is exactly the desired result. ☐

3.3. **Uniform continuity and asymptotic behavior.** A first idea to study the asymptotic behavior of integrals of Fourier transform of measures similar to the following one

$$\int_{\mathbb{R}^n} \frac{|\hat{\sigma}(x)|^2}{\|x-\xi\|^a} dx \ .$$

is naturally to study the asymptotic behavior at infinity of uniformly continuous functions.

Although as it is easy to see, the integrability of a function is not enough to ensure the limit of the function at infinity is zero, the next proposition shows that under the additional hypothesis of uniform continuity, the desired result is valid. This result could be taken in consideration when dealing with the Fourier transform of a measure which is a uniformly continuous function on $\mathbb{R}$. (see [16, p. 132]).

Proposition 4. *Let u be an integrable uniformly continuous function on $\mathbb{R}$. Then we have* $\lim_{|x|\to+\infty} u(x) = 0$.

Proof. By considering the positive and negative parts of u, we can restrain the proof to the case in which the function u is non negative. Furthermore, we will only consider the case of the limit in $+\infty$ as the proof is similar for the other case that is, the limit in $-\infty$. For $\delta > 0$, the integral of u can be decomposed in the following way:

$$\int_0^{+\infty} u(x)dx = \sum_{n=0}^{+\infty} \int_{n\delta}^{(n+1)\delta} u(x)dx < +\infty \ . \tag{3.25}$$

Using the mean value theorem of Cauchy (see [6, p. 321]), we get a real sequence $(x_n)_{n\in\mathbb{N}}$ such that:

$$\forall n \in \mathbb{N} \quad x_n \in [n\delta, (n+1)\delta] \ \ , \ \ u(x_n)\delta = \int_{n\delta}^{(n+1)\delta} u(x)dx \ ,$$

Observe that the decomposition of the integral, in (3.25), implies that $\lim_{n\to+\infty} u(x_n) = 0$ Now, for $\epsilon > 0$ given, we have, as a consequence of the uniform continuity of u that there exists $\delta_0 > 0$ such that:

$$\forall x, y \in \mathbb{R}_+ \ \ |x-y| \leq \delta_0 \ \Rightarrow \ |u(x) - u(y)| \leq \epsilon \ .$$

For any $x \in \mathbb{R}_+$, let n_0 be the integer such that $x \in [n_0\delta_0, (n_0+1)\delta_0[$. Then, for x big enough, such that $u(x_{n_0}) \leq \epsilon$, we get:

$$|u(x)| \leq |u(x) - u(x_{n_0})| + u(x_{n_0}) \leq 2\epsilon$$

and so, the result announced follows. □

Unfortunately, this kind of approach is doomed to fail as $\widehat{\sigma}$ need not go to zero at infinity for Radon measures σ, even for Radon measures with finite energy. See [25, p. 169] for positive partial results concerning the asymptotic behaviors of Fourier transforms of Radon measures.

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