

## Dynamical Value-at-Risk via Ito line integrals

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**Abstract.** Price and liquidity are both very important factors of risk in financial investments either stocks or bonds. We present a method to deal jointly with these two factors. We consider that return and volume of transactions, of a bond or stock, define a line in the plane which should be modelled by a stochastic line. We define risk of an investment as a stochastic Ito type line integral over the investment (return,volume) line. We present some properties of the stochastic line integral defined that are needed for future calculations or, interesting in themselves, and we apply these ideas to data from the portuguese stock exchange. We show that this model enables us to differentiate the risk of the stocks considered, taking in account not only price and its volatility, as usual, but also the dynamics of the volume of transactions.

**1. Introduction.** Most continuous time models describing the behavior of a stock in a financial market postulate an evolution law of the price of the stock as a stochastic function of time. When considering risk assessment related problems another possible approach is to model the time dependance of, both, price and volume of transactions. In doing so we will be giving price and liquidity - measured by volume of transactions - comparable roles as investment risk factors.

Stock or bond market liquidity, under the perspective adopted in this work, has been object of study in recent publications<sup>1</sup>. Liquidity, as a risk factor, may appear quantified not only as market volume of transactions, but also as bid-ask spread (as in [3] and [19]) or as absence of transaction costs (as in [10]) or even as the volatility in stock or bond price, as usually. A remarkable simulation study ([8]) offering an explanation of the smile pattern of implied volatility relating it to the lack of market liquidity shows the benefit of dealing jointly with price and volume. The next figure depicts, for a typical security, the evolution of both the price and the logarithm of volume of transactions.

In order to deal jointly with price and volume of transactions we will find useful to consider a sort of stochastic Ito line integral. This kind of integral was extensively studied by several authors (see [4],[25],[26],[27],[28] and [12]). The stochastic Ito line integral studied in this work has as advantages being simple to calculate with and invariant under a regular change of parameter.

Value-at-Risk (VaR) with all its disadvantages ([7]), still is a powerful and widespread used tool in financial risk management. After choosing an observable quantity - such as investment portfolio value (or returns) or credit portfolio value (or returns) - which is modelled by a random variable, say  $L$ , in order to define its associate VaR (see [13]), one looks at *the maximum loss with prescribed confidence level in a given time horizon*.

The random variable  $L$  is determined by data at a given date  $t$  and its VaR is obtained through quantile estimation. This quantile estimation can be done either in a classical - parametric or non parametric - way or using extreme value theory.

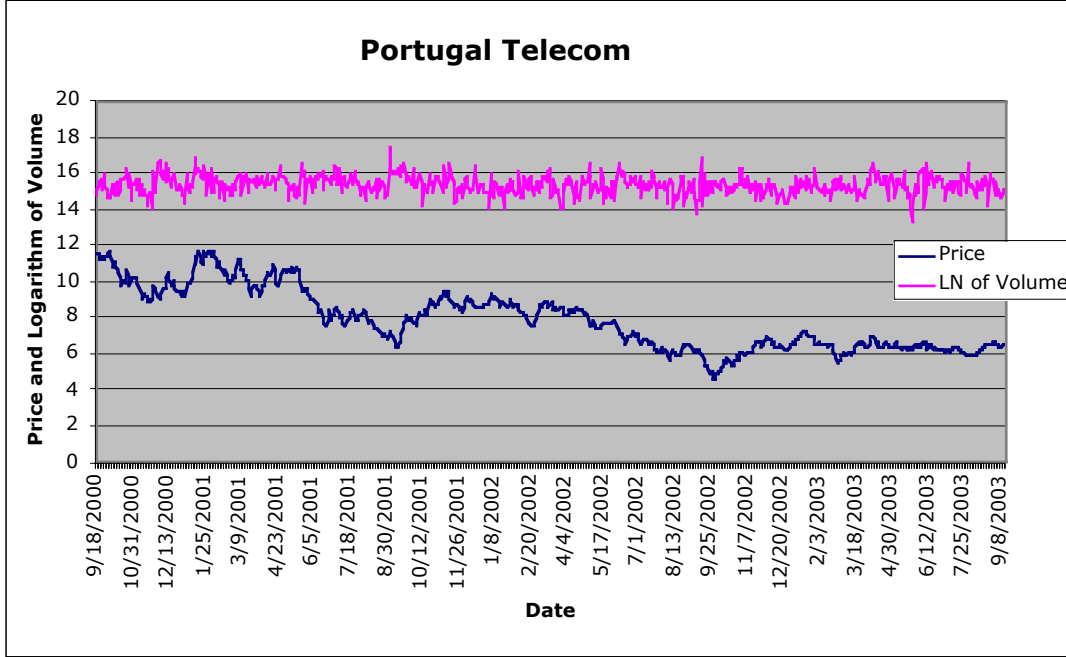
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<sup>1</sup>See, for instance at [http://www.defaultrisk.com/ps\\_liquidity.htm](http://www.defaultrisk.com/ps_liquidity.htm), the works [16],[1],[6],[24],[2],[21],[14].

A natural approach to Value-at-Risk is to think of the random quantity under study as a time dependent one, to model the time dependence of this quantity by means of a stochastic differential equation, to define the risk as a random variable given by a suitable norm of a trajectory of the solution and then, to estimate the tail of this random variable. We call this approach to Value-at-Risk, *dynamical value at risk* (D-VaR). The most important *a priori* advantage of D-VaR is that we have a structural model which embodies the desired characteristics of the evolution of the risk. Another advantage is that, in case of risky investments, the modelling of portfolio value is easily performed.



Let us observe first that the natural evolution law of some quantity thought to depend on both price and volume - such as the risk of a financial investment in the security - should be modelled by a stochastic line in the plane. The next figure shows the line defined by  $(\int_0^t \log(V_s) ds, \int_0^t r_s ds)$  where  $r_s$  stands for the return, and  $V_s$  for the volume of transactions for a typical security of the portuguese stock exchange Euronext Lisboa.

Suppose that we have for the price  $S_t$  and volume  $V_t$  of the security the following dynamics

$$\begin{cases} dS_t = \mu(t, S_t, V_t)dt + \sigma(t, S_t, V_t)dB_t^{(1)} & S_0 = S(0) \\ dV_t = \nu(t, S_t, V_t)dt + \rho(t, S_t, V_t)dB_t^{(2)} & V_0 = V(0) \end{cases} \quad (1)$$

with  $(B_t^{(1)})_{t \geq 0}$   $(B_t^{(2)})_{t \geq 0}$  brownian processes, possibly with a non trivial correlation structure. Then the dynamics of the line  $\gamma(t)$  given for  $t \geq 0$  by

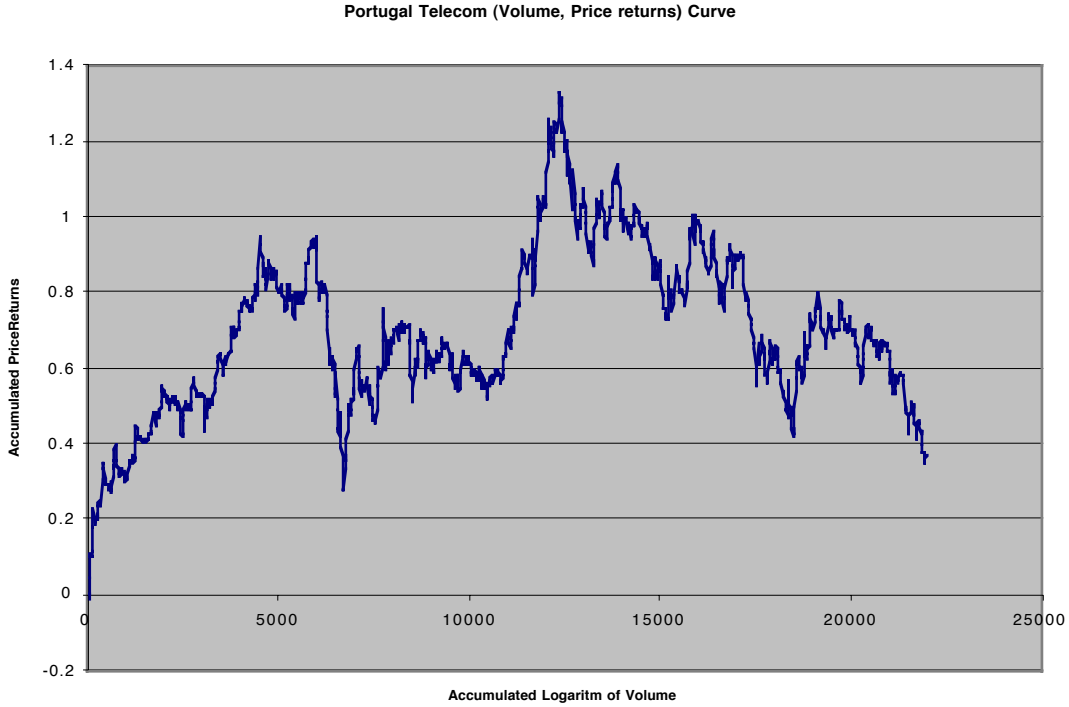
$$\gamma(t) = \left( \int_0^t V_s ds, \int_0^t r_s ds \right), \quad (2)$$

is perfectly known.

Observing that in the plane where  $\gamma(t)$ , as defined by 2, takes its values the upper right region is a lesser riskier region and the lower left region is a more riskier region it is natural to postulate that he evolution of the risk of some investment is given by a solution  $X$  of a stochastic differential equation given by:

$$X_z = X_0 + \int_{\gamma} f(X_w) dw + \int_{\gamma} g(X_w) dB_w, \quad (3)$$

with  $z$  in the plane.



The integral on the left is an usual line integral and the integral on the right should be a sort of a stochastic Ito like line integral with respect to some brownian process. In the following we summarize some of the properties of a stochastic line Ito integral we get from a natural definition of this integral and, then, we apply this formalism to data from the portuguese stock exchange. We will try to show that even in the case of a very simple model a discrimination of stocks with respect to risk, having as factors price and volume, can be achieved.

**2. The Stochastic Line Ito Integral.** In what follows we consider a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , brownian process  $(B_t)_{t \in [0, +\infty[}$  defined in it and the associated brownian filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ . Let  $f$  be a stochastic process with state space  $\mathbb{C}$  satisfying some regularity conditions which will be made precise later.

Let us consider a stochastic line in the plane  $\gamma$  which is, almost surely, a continuous map, piecewise differentiable with a continuous derivative, defined on  $[a, b]$  taking values in the plane  $\mathbb{C}$  and an adapted processes with respect to the brownian filtration  $\mathbb{F}$ . Denote by  $\gamma^*$  the image of the line, that is  $\gamma^* := \gamma([a, b])$ . Let  $A = \gamma(a)$  and  $B = \gamma(b)$  be, respectively, the initial point and the final point of  $\gamma$ . Recall that  $L(f^2, \gamma)$  defined as the length of the line in  $\mathbb{C}$  given by the map  $f^2 \circ \gamma$  with domain in  $[a, b]$  is by definition, almost surely,

$$L(f^2, \gamma) := \int_{\gamma} |f^2(z)| |dz| = \int_a^b |f(\gamma(t))|^2 |\gamma'(t)| dt .$$

A natural condition to have for the stochastic line integral is that the Ito isometry property - in a generalized way - still holds as, for instance in the next formula:

$$\mathbb{E} \left[ \left| \int_{\gamma} f(z) dB_z \right|^2 \right] = \mathbb{E} \left[ \int_{\gamma} |f^2(z)| |dz| \right] = \mathbb{E} [L(f^2, \gamma)] . \quad (4)$$

**Definition 2.1.** Let  $f$  be  $\mathbb{F}$  adapted and such that

$$\mathbb{P} \left[ \int_{\gamma} |f^2(z)| |dz| < +\infty \right] = 1 .$$

The **brownian line integral** of  $f$  over the line  $\gamma$  is defined by:

$$\int_{\gamma} f(z)dB_z := \int_a^b f(\gamma(t)) \cdot \sqrt{|\gamma'(t)|} \cdot \exp(i \frac{\text{Arg } \gamma'(t)}{2}) dB_t. \quad (5)$$

We are considering in fact a determination of the (complex) square root of  $\gamma'$  which is defined and continuous in  $\mathbb{C} - \{0\}$ . We will suppose, in all of the following that  $\gamma' \neq 0$  almost surely. As we will see this is not a limitation for the applications.

**Remark 2.2.** With the previous definition, if  $\mathbb{E} \left[ \int_{\gamma} |f^2(z)| |dz| \right] < +\infty$  we have by Ito's isometry property:

$$\begin{aligned} \mathbb{E} \left[ \left| \int_{\gamma} f(z)dB_z \right|^2 \right] &= \mathbb{E} \left[ \left| \int_a^b f(\gamma(t)) \cdot \sqrt{|\gamma'(t)|} \cdot e^{i \frac{\text{Arg } \gamma'(t)}{2}} dB_t \right|^2 \right] = \\ &= \mathbb{E} \left[ \int_a^b |f(\gamma(t))|^2 |\gamma'(t)| dt \right] \end{aligned}$$

which verifies the condition 4.

As an important property, the integral defined by 5 is invariant by a random change of parametrization of the stochastic line  $\gamma$ . The sense in which invariance is meant is conveyed by the next example.

**Example 2.3.** Consider the line  $\gamma$  defining the real interval  $[0, a]$  in  $\mathbb{R}_+$  and two different parametrizations of this line given by  $\gamma(t) = t$  for  $t \in [0, a]$  and  $\gamma_1(t) = ta$  for  $t \in [0, 1]$ . Then for  $f \equiv 1$ ,

$$\int_{\gamma} f(z)dB_z = \int_0^a dB_t = B_a$$

and

$$\int_{\gamma_1} f(z)dB_z = \int_0^1 a dB_t = aB_1.$$

Observe that the law of the random variable  $B_a$  is equal to the law of  $aB_1$ , that is  $\mathcal{N}(0, \sqrt{a})$ .

A general result in this line of thought is the following.

**Proposition 2.4. (Parametrization invariance)** Let  $\psi$  be, almost surely, a  $C^1$  one to one map from  $[c, d]$  onto  $[a, b]$ , adapted to the brownian filtration, such that  $\psi(c) = a$  and  $\psi(d) = b$ . Then  $\gamma_1 := \gamma \circ \psi$  is another stochastic line, defined in  $[c, d]$ , but with same initial and final points as  $\gamma$  and

$$\int_{\gamma_1} f(z)dB_z = \int_{\gamma} f(z)dB_z \text{ in law.}$$

**Proof.** This result is a consequence of time change results in brownian processes. By definition 5 and observing that the conditions imposed on  $\psi$  imply that  $\psi' > 0$  almost surely, we have

$$\begin{aligned} \int_{\gamma_1} f(z)dB_z &= \int_c^d f(\gamma_1(t)) \sqrt{|\gamma_1'(t)|} e^{i \frac{\text{Arg } \gamma_1'(t)}{2}} dB_t = \\ &= \int_c^d f(\gamma(\psi(t))) \sqrt{|\gamma'(\psi(t))\psi'(t)|} \cdot e^{i \frac{\text{Arg}(\gamma'(\psi(t)) \times \psi'(t))}{2}} dB_t = \\ &= \int_c^d f(\gamma(\psi(t))) \sqrt{|\gamma'(\psi(t))\psi'(t)|} \cdot e^{i \frac{\text{Arg} \gamma'(\psi(t))}{2}} \cdot e^{i \frac{\text{Arg} \psi'(t)}{2}} dB_t = \\ &= \int_c^d f(\gamma(\psi(t))) \sqrt{|\gamma'(\psi(t))\psi'(t)|} \cdot e^{i \frac{\text{Arg} \gamma'(\psi(t))}{2}} dB_t. \end{aligned}$$

By the change of variable  $\psi(t) = u$  and  $t = \psi^{-1}(u)$  and as  $a = \psi(c)$  and  $b = \psi(d)$ , we get:

$$\int_{\gamma_1} f(z)dB_z = \int_a^b f(\gamma(u))\sqrt{|\gamma'(u)|}\sqrt{|\psi'(\psi^{-1}(u))|} \cdot e^{i\frac{\text{Arg}\gamma'(u)}{2}} dB_{\psi^{-1}(u)} \quad (6)$$

Now, taking in account the hypotheses, observe that as:

$$\psi(t) = a + \int_c^t \psi'(s)ds$$

all the conditions of standard results of time change in brownian motion are satisfied (see, for instance, corollary 8.5.5 in [17][p. 147]). So, defining the process

$$Y_t := \int_a^t \sqrt{\psi'(s)}dB_s ,$$

the process  $\tilde{B}_t := Y_{\psi^{-1}(t)}$  is a brownian process for  $t \in [a, b]$ . By a standard result, of integration with respect to an integral process as  $dY_t = \sqrt{\psi'(t)}dB_t$ , we have:

$$d\tilde{B}_u = dY_{\psi^{-1}(u)} = \sqrt{\psi'(\psi^{-1}(u))}dB_{\psi^{-1}(u)} ,$$

and so, using this expression in the integral of the right side of formula 6, we have

$$\int_{\gamma_1} f(z)dB_z = \int_a^b f(\gamma(u))\sqrt{|\gamma'(u)|} \cdot e^{i\frac{\text{Arg}\gamma'(u)}{2}} d\tilde{B}_u ,$$

and the result is proved. ■

Let us observe by some examples the general outcome of a line integral of this kind.

**Example 2.5.** Let  $f \equiv 1$  and  $\gamma$  given, for  $t \in [0, 1]$  by  $\gamma(t) = tz_2 + (1-t)z_1$ , that is  $\gamma^* = [z_1, z_2]$  the segment in the plane defined by the two points  $z_1, z_2 \in \mathbb{C}$ . Then if  $z_t = tz_2 + (1-t)z_1$  is the running point in  $\gamma^*$ ,

$$F(z_t) := \int_{\gamma(t)} f(z)dB_z = \sqrt{|z_2 - z_1|} e^{i\frac{\text{Arg}(z_2 - z_1)}{2}} B_t = \sqrt{|z_2 - z_1|} e^{i\frac{\text{Arg}(z_2 - z_1)}{2}} Z_{(0, \sqrt{t})} .$$

where  $Z_{(0, \sqrt{t})} \in \mathcal{N}(0, \sqrt{t})$ .

**Example 2.6.** Let, again,  $f \equiv 1$  and  $\gamma^* = \partial B(0, R)$  the sphere of radius  $R$  in the plane, that is such that:  $\gamma(t) = Re^{2\pi it}$  for  $t \in [0, 1]$ . Then:

$$\begin{aligned} \int_{\gamma} f(z)dB_z &= \sqrt{2\pi R} e^{i\frac{\pi}{4}} \int_0^1 e^{\pi it} dB_t = \sqrt{2\pi R} (1+i) \left( \int_0^1 \cos(\pi t) dB_t + i \int_0^1 \sin(\pi t) dB_t \right) = \\ &= \sqrt{2\pi R} (1+i) \left( Z_{(0, \frac{1}{\sqrt{2}})}^1 + i Z_{(0, \frac{1}{\sqrt{2}})}^2 \right) , \end{aligned}$$

where for  $k = 1, 2$ , we have  $Z_{(0, \frac{1}{\sqrt{2}})}^k \in \mathcal{N}(0, \frac{1}{\sqrt{2}})$ .

**2.1. Martingale property.** In order to obtain, in a natural way, a martingale property for the stochastic line integral we consider the integral as a function of the running point of the line. Let  $\gamma$  be a line defined over the interval  $[a, b]$ . Let, for all  $t$  in  $[a, b]$ , the running point of  $\gamma$  be denoted by  $z = z_t = \gamma(t)$  and  $A = z_a$  and  $B = z_b$  the the initial and the final points of  $\gamma$ .

Consider the natural order over  $\gamma^*$  induced by the order over  $[a, b]$  and defined by:

$$\forall z_r, z_s \in \gamma^* \quad z_r \preceq z_s \Leftrightarrow r \leq s .$$

Associated to this order one can consider  $\mathbb{F}^\gamma = (\mathcal{F}_z)_{z \in \gamma^*}$  a brownian filtration over  $\gamma^*$  defined by:

$$\forall z \in \gamma^*, z = z_s s \in [a, b] \quad \mathcal{F}_z := \mathcal{F}_s .$$

An  $\mathbb{F}^\gamma$  adapted process  $(M_z)_{z \in \gamma^*}$ , of integrable random variables is a martingale if

$$\forall z, w \in \gamma^* \quad z \preceq w \Rightarrow \mathbb{E}[M_w \mid \mathcal{F}_z] = M_z .$$

**Proposition 2.7.** *Let  $\gamma$  be a line in  $\mathbb{C}$  and  $f$  such that  $\gamma' \in L^\infty(\Omega \times [a, b])$  and  $f$  a  $\mathbb{F}^\gamma$  adapted stochastic process defined on  $\gamma^*$  such that  $f \circ \gamma \in L^2(\Omega \times [a, b])$ . Then*

$$M_z := \int_{\gamma(A, z)} f(w) dB_w ,$$

is an  $\mathbb{F}^\gamma$  martingale and the usual Doob's inequalities are verified, that is:

$$\forall p > 1 \quad \forall \lambda > 0 \quad \mathbb{P} \left[ \sup_{z \in \gamma^*} \left| \int_{\gamma(A, z)} f(w) dB_w \right| \geq \lambda \right] \leq \frac{1}{\lambda^p} \mathbb{E} \left[ \left| \int_{\gamma(A, B)} f(w) dB_w \right|^p \right] \quad (7)$$

and

$$\mathbb{E} \left[ \sup_{z \in \gamma^*(A, B)} \left| \int_{\gamma(A, z)} f(w) dB_w \right|^2 \right] \leq 4 \mathbb{E} \left[ \left| \int_{\gamma(A, B)} f(w) dB_w \right|^2 \right] = 4 \mathbb{E} [L(f^2, \gamma(A, B))]$$

**Proof.** It is a straightforward consequence of the definitions and of the usual Doob's inequality.  $\blacksquare$

**2.2. Analytic functions defined by the stochastic line integral.** The stochastic line integral under study enjoys an interesting property described in the next result.

**Theorem 2.8.** *Let  $\gamma$  be a line in  $\mathbb{C}$  with parameter space the interval  $[a, b]$ , and  $(\psi_w)_{w \in \gamma^*}$  and  $(\phi_w)_{w \in \gamma^*}$  be stochastic processes adapted to  $(\mathcal{F}_w)_{w \in \gamma^*}$  such that:*

1.  $\|\psi\|_{L^\infty(\Omega \times \gamma^*)} < +\infty$ ;
2. there exists  $K$ , a non random compact set in  $\mathbb{C}$  such that  $\phi(\gamma^*) \subset K$  almost surely.

Then,  $F$  defined by

$$\forall z \in K^c \quad F(z) := \int_{\gamma(A, B)} \frac{\psi_w}{\phi_w - z} dB_w ,$$

is, almost surely, an analytical function.

**Proof.** Following a standard argument (see [23][p. 200]), consider an arbitrary point  $a \in K^c$  and  $r_a > 0$  such that,  $B(a, r_a)$ , the open ball in  $\mathbb{C}$  centered in  $a$  with radius  $r_a$ , satisfies  $B(a, r_a) \subset K^c$ . Then, observe that, by simple calculation,

$$\forall z \in B(a, r_a) \quad \frac{1}{\phi_w - z} = \sum_{n=0}^{+\infty} \frac{(z - a)^n}{(\phi_w - a)^{n+1}} . \quad (8)$$

Consider now the process  $f$  defined by:

$$\forall z \in B(a, r_a) \quad f(z) := \sum_{n=0}^{+\infty} \left( \int_{\gamma(A, B)} \frac{\psi_w}{(\phi_w - a)^{n+1}} dB_w \right) (z - a)^n$$

We will show, first, that  $f$  defines, almost surely, an analytical function on  $K^c$  and then, that  $f(z) = F(z)$  almost surely for every  $z \in K^c$ . For each  $n \in \mathbb{N}$ , consider the random variable  $X_n$  defined by:

$$X_n := \int_{\gamma(A,B)} \frac{\psi_w}{(\phi_w - a)^{n+1}} dB_w$$

and observe that by the hypotheses assumed and Ito's inequality (see, for instance, [15][p. 25])  $\mathbb{E}[X_n] = 0$  and

$$\mathbb{E}[|X_n|^2] \leq \|\psi\|_\infty \mathbb{E} \left[ \int_a^b \frac{|\gamma'(t)|}{|\phi(\gamma(t)) - a|^{2n+2}} dt \right] \leq \frac{\|\psi\|_\infty}{r_a^{2n+2}} \mathbb{E}[L(\gamma(A, B))] . \quad (9)$$

With the notation  $z - a = re^{ix}$  we have  $f(z) = \sum_{n=0}^{+\infty} (X_n r^n) e^{inx}$ . We will study the convergence of this random trigonometric series or, for a start, the convergence of the non random trigonometric series

$$\sum_{n=0}^{+\infty} (\mathbb{E}[|X_n|] r^n) e^{inx} .$$

For a fixed  $z_0 \in B(a, r_a)$ , with  $|z_0 - a| = r_0 < r_a$  we have by Jensen inequality and inequality given in formula 9:

$$(\mathbb{E}[|X_n|] r_0^n)^2 \leq \mathbb{E}[|X_n|^2] r_0^{2n} \leq \frac{\|\psi\|_\infty}{r_a^2} \mathbb{E}[L(\gamma(A, B))] \left(\frac{r_0}{r_a}\right)^{2n} ,$$

thus showing that  $\sum_{n=0}^{+\infty} (\mathbb{E}[|X_n|] r_0^n)^2 < +\infty$ . By Riesz-Fisher theorem there is  $g$  an  $L^2([0, 1])$  function such that  $g = \sum_{n=0}^{+\infty} (\mathbb{E}[|X_n|] r_0^n) e^{inx}$ . Now, by Carleson's theorem (see [5] or [22]), this series converges almost everywhere in  $x \in [0, 1]$  and so, for at least an  $x_0 \in [0, 1]$ , it converges implying that for  $z_0 - a := r_0 e^{ix_0} \in B(a, r_a)$  the numerical series

$$\sum_{n=0}^{+\infty} \mathbb{E}[|X_n|] (z_0 - a)^n$$

converges. Now by Abel's lemma the series converges in the whole ball  $B(a, r_0)$  and, as  $r_0 < r_a$  was arbitrary, the series converges in  $B(a, r_a)$ . So the series given by  $\sum_{n=0}^{+\infty} \mathbb{E}[|X_n|] (z - a)^n$  converges absolutely for  $z \in B(a, r_0)$ , that is:

$$\mathbb{E} \left[ \sum_{n=0}^{+\infty} |X_n| |z - a|^n \right] < +\infty$$

and so, almost surely,

$$\sum_{n=0}^{+\infty} |X_n| |z - a|^n < +\infty ,$$

showing that  $\sum_{n=0}^{+\infty} X_n (z - a)^n$  defines  $f$  as an analytical function almost surely. We will now prove that  $f = F$  almost surely. Consider, again, the development valid almost surely for  $z \in B(a, r_a)$  given in formula 8. Then, with the usual interpretation, we have that:

$$\begin{aligned} F(z) &= \int_{\gamma(A,B)} \left( \sum_{n=0}^{+\infty} \frac{\psi_w}{(\phi_w - a)^{n+1}} (z - a)^n \right) dB_w = \\ &= \int_a^b \left( \sum_{n=0}^{+\infty} \frac{\psi_{\gamma(t)}}{(\phi_{\gamma(t)} - a)^{n+1}} (z - a)^n \right) \sqrt{|\gamma'(t)|} e^{i \frac{\text{Arg} \gamma'(t)}{2}} dB_t . \end{aligned} \quad (10)$$

Now observe that, defining

$$G_n(t) = \frac{\psi_{\gamma(t)}}{(\phi_{\gamma(t)} - a)^{n+1}} (z - a)^n \sqrt{|\gamma'(t)|} e^{i \frac{\text{Arg} \gamma'(t)}{2}},$$

we have in  $L^2(\Omega \times [a, b])$  that

$$\|G_n\|_2 \leq \frac{|z - a|^{2n}}{r_a^{2n+2}} \|\psi\|_\infty \mathbb{E}[L(\gamma(A, B))],$$

showing that the series  $\sum_{n=0}^{+\infty} \|G_n\|_2$  converges for every  $z \in B(a, r_a)$ . This implies, in particular, that the series  $\sum_{n=0}^{+\infty} G_n$  converges in  $L^2(\Omega \times [a, b])$  for every  $z \in B(a, r_a)$ , but more can be said. In fact, let  $S_N$  be the partial sum of order  $N$  of the series in the right hand side of the first line of formula 10 and  $S$  be its sum. Then, again, by Ito's inequality:

$$\begin{aligned} \left\| \int_{\gamma(A, B)} S_N(w) dB_w - \int_{\gamma(A, B)} S(w) dB_w \right\|_{L^2(\Omega)} &= \mathbb{E} \left[ \left| \int_{\gamma(A, B)} (S_N(w) - S(w)) dB_w \right|^2 \right] \leq \\ &\leq \mathbb{E} \left[ \int_a^b |S_N(w) - S(w)|^2 |dw| \right] = \\ &= \|S_N - S\|_{L^2(\Omega \times \gamma^*)} \leq \sum_{n=N+1}^{+\infty} \|G_n\|_{L^2(\Omega \times [a, b])}, \end{aligned}$$

thus showing that, with the limits being taken in  $L^2(\Omega)$ ,

$$F(z) = \int_{\gamma(A, B)} \left( \lim_{N \rightarrow +\infty} S_N(w) \right) dB_w = \lim_{N \rightarrow +\infty} \left( \int_{\gamma(A, B)} S_N(w) dB_w \right) = f(z)$$

which shows that  $f = F$  in  $L^2(\Omega)$  which, finally, implies the equality, almost surely, stated.  $\blacksquare$

**2.3. Ito Formula.** Let  $\gamma$  be a random line in  $\mathbb{C}$ , with parameter domain  $[a, b]$ , and  $z = z_t = \gamma(t)$ , for  $t \in [a, b]$  the running point of  $\gamma^*$ . Let  $\gamma_{(z_a, z_t)} = \gamma|_{[a, t]}$  be the restriction of  $\gamma$  to  $[a, t]$ . Let  $A := \gamma(a)$ .

**Definition 2.9. (Ito Process)** An Ito Line Process is a process given by

$$Y_z = Y_A + \int_{\gamma} f(z) dz + \int_{\gamma} g(z) dB_z, \quad (11)$$

which is interpreted by considering  $X_t = Y_{\gamma(t)}$ ,  $F(t) := f(\gamma(t))$  and  $G(t) := g(\gamma(t))$  as

$$X_t = X_a + \int_a^t F(s) \gamma'(s) ds + \int_a^t G(s) \sqrt{|\gamma'(s)|} e^{i \frac{\text{Arg} \gamma'(s)}{2}} dB_s.$$

**Remark 2.10.** As usual, formula 11 can also be represented by the following formula,

$$d_{\gamma} Y_z = f(z) d_{\gamma} z + g(z) d_{\gamma} B_z, \quad (12)$$

where the subscript  $\gamma$  indicates the line dependence of the integral and may be omitted, if no confusion is anticipated.

The following lemma shows how an  $C^2$  function acts upon a Ito line process.

**Lemma 2.11.** Let  $h(z, w)$  be a  $C^2(\mathbb{C} \times \mathbb{C})$  function and consider the image process of  $Y$  by  $h$ , given by  $Z_z = h(z, Y_z)$ . Then  $Z$  we have, with the obvious interpretation as an integral and with the line dependence omitted,

$$\begin{aligned} dZ_z &= \left( \frac{\partial h}{\partial z} dz + \frac{\partial h}{\partial \bar{z}} d\bar{z} \right) + \left( \frac{\partial h}{\partial w} dY_z + \frac{\partial h}{\partial \bar{w}} d\bar{Y}_z \right) + \\ &+ \left( \frac{1}{2} \frac{\partial^2 h}{\partial w^2} (dY_z)^2 + \frac{\partial^2 h}{\partial w \partial \bar{w}} dY_z d\bar{Y}_z + \frac{1}{2} \frac{\partial^2 h}{\partial \bar{w}^2} (d\bar{Y}_z)^2 \right). \end{aligned} \quad (13)$$



**Proof.** This lemma is a consequence of the multidimensional Ito's Formula as given, for instance, in [17][p. 49]. Take an usual vectorial Ito process  $X$ , with  $m$  components, a function of class  $C^2$ ,  $h(t_1, \dots, t_n, x_1, \dots, x_m)$  and the process  $Y = Y_{(t_1, \dots, t_n)}$  defined by  $Y_{(t_1, \dots, t_n)} = h(t_1, \dots, t_n, X_1, \dots, X_m)$ . Then, by Ito's formula, we have:

$$dY = \sum_{i=1}^n \frac{\partial h}{\partial t_i} dt_i + \sum_{i=1}^m \frac{\partial h}{\partial x_i} dX_i + \frac{1}{2} \sum_{i,j=1}^{n,m} \frac{\partial^2 h}{\partial x_i \partial x_j} dX_i dX_j .$$

In fact, by applying this multidimensional Ito's formula to the decomposition of  $h$  in its real and imaginary parts  $h = h_1 + ih_2$ , and considering that for  $j = 1, 2$ ,  $h_j = h_j(t_1, t_2, x_1, x_2)$ , then  $W_t := Z_{\gamma(t)} = h(\gamma_1(t), \gamma_2(t), X_t^1, X_t^2)$  is an usual Ito process given by,

$$\begin{aligned} dW_t = & \left( \left[ \frac{\partial h_1}{\partial t_1} \gamma_1'(t) + \frac{\partial h_1}{\partial t_2} \gamma_2'(t) \right] + i \left[ \frac{\partial h_2}{\partial t_1} \gamma_1'(t) + \frac{\partial h_2}{\partial t_2} \gamma_2'(t) \right] \right) dt + \\ & + \left[ \left( \frac{\partial h_1}{\partial x_1} dX_1 + \frac{\partial h_1}{\partial x_2} dX_2 \right) + i \left( \frac{\partial h_2}{\partial x_1} dX_1 + \frac{\partial h_2}{\partial x_2} dX_2 \right) \right] + \\ & + \left( \frac{\partial^2 h_1}{\partial x_1 \partial x_2} + i \frac{\partial^2 h_2}{\partial x_1 \partial x_2} \right) dX_1 dX_2 + \\ & + \frac{1}{2} \left( \frac{\partial^2 h_1}{\partial x_1^2} + i \frac{\partial^2 h_2}{\partial x_1^2} \right) (dX_1)^2 + \frac{1}{2} \left( \frac{\partial^2 h_1}{\partial x_2^2} + i \frac{\partial^2 h_2}{\partial x_2^2} \right) (dX_2)^2 . \end{aligned} \quad (14)$$

Then, by considering the classical (see [11][p. 1, 22] or [9][p. 4]), holomorphic operators,

$$\frac{\partial}{\partial w} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{w}} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right)$$

and the complex conjugated differential forms,

$$dw = dx_1 + i dx_2 \quad \text{and} \quad d\bar{w} = dx_1 - i dx_2 ,$$

we have, for instance, that:

$$\begin{aligned} d_w h & := \left( \frac{\partial h}{\partial w} dY_z + \frac{\partial h}{\partial \bar{w}} d\bar{Y}_z \right) = \\ & = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) (h_1 + ih_2) (dX_1 + idX_2) + \\ & + \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) (h_1 + ih_2) (dX_1 - idX_2) = \\ & = \left[ \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) (h_1 + ih_2) + \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) (h_1 + ih_2) \right] dX_1 + \\ & + i \left[ \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) (h_1 + ih_2) - \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) (h_1 + ih_2) \right] dX_2 = \\ & = \left( \frac{\partial h_1}{\partial x_1} + i \frac{\partial h_2}{\partial x_1} \right) dX_1 + \left( \frac{\partial h_1}{\partial x_2} + i \frac{\partial h_2}{\partial x_2} \right) dX_2 = \\ & = \left( \frac{\partial h_1}{\partial x_1} dX_1 + \frac{\partial h_1}{\partial x_2} dX_2 \right) + i \left( \frac{\partial h_2}{\partial x_1} dX_1 + \frac{\partial h_2}{\partial x_2} dX_2 \right) , \end{aligned}$$

thus showing that  $d_w h$ , defined to be the second term on the right-hand side of formula 13 is equal to the terms on the second line of the right hand side of formula 14. The same reasoning applies to the other partial derivative of the first order by considering that if  $X_1 = \gamma_1$  and  $X_2 = \gamma_2$  then  $dX_1 = \gamma_1' dt$  and  $dX_2 = \gamma_2' dt$ , thus showing that the first terms on the right hand side of formulas 13 and 14 are

equal. As for the derivative of second order we have that:

$$\begin{aligned}
 d_w^2 h &= \left( \frac{1}{2} \frac{\partial^2 h}{\partial w^2} (dY_z)^2 + \frac{\partial^2 h}{\partial w \partial \bar{w}} dY_z d\bar{Y}_z + \frac{1}{2} \frac{\partial^2 h}{\partial \bar{w}^2} (d\bar{Y}_z)^2 \right) \\
 &= \frac{1}{8} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right)^2 h (dX_1 + i dX_2)^2 + \\
 &+ \frac{1}{4} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) h (dX_1 + i dX_2) (dX_1 - i dX_2) + \\
 &+ \frac{1}{8} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right)^2 h (dX_1 - i dX_2)^2 .
 \end{aligned}$$

Now by expanding the squares and the products we have,

$$\begin{aligned}
 d_w^2 h &:= \frac{1}{8} \left( \frac{\partial^2}{\partial x_1^2} - 2i \frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_2^2} \right) h ((dX_1)^2 + 2idX_1 dX_2 - (dX_2)^2) + \\
 &+ \frac{1}{4} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) h ((dX_1)^2 + (dX_2)^2) + \\
 &+ \frac{1}{8} \left( \frac{\partial^2}{\partial x_1^2} + 2i \frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_2^2} \right) h ((dX_1)^2 - 2idX_1 dX_2 - (dX_2)^2) = \\
 &= \frac{1}{2} \frac{\partial^2}{\partial x_1^2} h (dX_1)^2 + \frac{\partial^2}{\partial x_1 \partial x_2} h + \frac{1}{2} \frac{\partial^2}{\partial x_2^2} h (dX_2)^2
 \end{aligned}$$

This, finally, shows that  $d_w^2 h$  defined to be the third term on the right hand side of formula 13 is equal to the sum of the third and fourth terms on the right hand side of formula 14.  $\blacksquare$

As a consequence of this lemma we have an Ito formula for Ito line integrals.

**Theorem 2.12. (Ito Formula for line integrals)** *Let  $h(z, w)$  be a  $C^2(\mathbb{C} \times \mathbb{C})$  function, holomorphic in each variable and consider the image process of  $Y$  by  $h$ , given by  $Z_z = h(z, Y_z)$ . Then  $Z$  is also an Ito line process and we have, with the usual interpretation in integral form,*

$$d_\gamma Z_z = \frac{\partial h}{\partial z} d_\gamma z + \frac{\partial h}{\partial w} d_\gamma Y_z + \frac{1}{2} \frac{\partial^2 h}{\partial w^2} (d_\gamma Y_z)^2 . \quad (15)$$

**Proof.** Formula 15 is a straightforward consequence of formula 13 in the case of holomorphic functions as the conjugated terms disappear. In fact we have as a consequence of the holomorphic character of  $h$  that

$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial h}{\partial \bar{w}} = \frac{\partial^2 h}{\partial w \partial \bar{w}} = \frac{\partial^2 h}{\partial \bar{w}^2} = 0 ,$$

thus implying that:

$$d_\gamma Z_z = \frac{\partial h}{\partial z} d_\gamma z + \frac{\partial h}{\partial w} d_\gamma Y_z + \frac{1}{2} \frac{\partial^2 h}{\partial w^2} (d_\gamma Y_z)^2 .$$

Now, using the usual interpretation we have:

$$d_\gamma Y_z = f d_\gamma z + g d_\gamma B_z$$

and

$$(d_\gamma Y_z)^2 = g^2 d_\gamma z ,$$

we get, as a consequence, that formula 15 can be read as:

$$d_\gamma Z_z = \left[ \frac{\partial h}{\partial z} + f \frac{\partial h}{\partial w} + \frac{1}{2} g^2 \frac{\partial^2 h}{\partial w^2} \right] d_\gamma z + g \frac{\partial h}{\partial w} d_\gamma B_z , \quad (16)$$

which is an usual Ito Formula and shows that  $Z$  is an Ito line process.  $\blacksquare$

**Remark 2.13.** *The definition of an Ito line process – given above as formula 11 and its interpretation formula 12 – is a bona fide generalization of the usual Ito process definition (see [17][p. 44]). With the notations of given above suppose that  $f$  and  $g$  are real and that  $\gamma^* \subset \mathbb{R}$ . Due to reparametrization invariance – proposition 2.4. – we can suppose that  $\gamma^* = [0, T]$ , for some  $T$ , and that  $\gamma(t) = t$ . Then, if*

$$Y_z = Y_A + \int_{\gamma} f(z) dz + \int_{\gamma} g(z) dB_z ,$$

the usual interpretation gives

$$\begin{aligned} Y_{\gamma(t)} &= Y_{\gamma(0)} + \int_0^t f(\gamma(s)) \gamma'(s) ds + \int_a^t g(\gamma(s)) \sqrt{|\gamma'(s)|} e^{i \frac{\text{Arg} \gamma'(s)}{2}} dB_s \\ &= Y_0 + \int_0^t f(s) ds + \int_a^t g(s) dB_s = Y(t) . \end{aligned}$$

**2.4. Some Stochastic Differential Equations.** The integral of the first kind is essentially a one dimensional integral. Standard results of the theory of stochastic differential equations on the line immediately apply. Let  $\gamma$  be a random line in  $\mathbb{C}$ , with parameter domain  $[a, b]$ , and  $z = z_t = \gamma(t)$ , for  $t \in [a, b]$  the running point of  $\gamma^*$ . Let  $\gamma_{(z_a, z)} = \gamma_{(z_a, z_t)}$  be the restriction of  $\gamma$  to  $[a, t]$ . Recall that  $A := \gamma(a)$  and denote by

$$Y_z = Y_A + \int_{\gamma_{(z_a, z)}} f(z, Y_z) dz + \int_{\gamma_{(z_a, z)}} g(z, Y_z) dB_z , \quad (17)$$

the stochastic differential equation in the line, which when  $X_t = Y_{z_t} = Y_{\gamma(t)}$ , is given by:

$$X_t = X_a + \int_a^t f(\gamma(s), X_s) \cdot \gamma'(s) ds + \int_a^t g(\gamma(s), X_s) \sqrt{|\gamma'(s)|} \cdot e^{i \frac{\text{Arg}(\gamma'(s))}{2}} dB_s . \quad (18)$$

Under appropriate hypotheses on  $f, g$  and  $\gamma$  the existence of an unique strong solution can be assured.

**Example 2.14.** *A simple application of theorem 2.12., that is Ito formula for line integrals, shows that the stochastic differential equation*

$$Y_z = Y_A + \int_{\gamma_{(z_a, z)}} \mu_w Y_w dw + \int_{\gamma_{(z_a, z)}} \sigma_w Y_w dB_w , \quad (19)$$

which can also be written as

$$d_{\gamma} Y_z = \mu_z Y_z d_{\gamma} z + \sigma_z Y_z d_{\gamma} B_z ,$$

has the process

$$Y_z = Y_A \exp \left( -\frac{1}{2} \int_{\gamma_{(z_a, z)}} (\mu_w - \frac{1}{2} \sigma_w^2) dw + \int_{\gamma_{(z_a, z)}} \sigma_w dB_w \right) .$$

as a solution. In fact, a simple calculation with  $dY_z = (\mu_z - \sigma_z^2/2) d_{\gamma} z + \sigma_z d_{\gamma} B_z$  and taking  $h(z, w) = \exp(w)$  and  $Z_z = h(z, Y_z)$  in formula 16, give  $d_{\gamma} Z_z = \mu_z Z_z d_{\gamma} z + \sigma_z Z_z d_{\gamma} B_z$  which is, exactly, equation 19.

Observe that if,  $\mu_z \equiv 0$ ,  $\sigma_z \equiv 1$  then equation 19 is replaced by:

$$Y_z = Y_A + \int_{\gamma_{(z_a, z)}} Y_z dB_z ,$$

having as a solution,

$$Y_z = Y_A \exp \left( -\frac{1}{2} [z - A] + \int_{\gamma_{(z_a, z)}} dB_z \right) .$$

If now,  $\gamma$  is a closed line, that is  $A = \gamma(a) = \gamma(b) = B$ , then,

$$Y_B = Y_A \exp \left( \int_{\gamma(A,B)} dB_z \right),$$

where  $\int_{\gamma(A,B)} dB_z = \int_a^b \sqrt{|\gamma'(t)|} \cdot e^{i \frac{\text{Arg}(\gamma'(t))}{2}} dB_t$ , is a random variable with mean zero and variance bounded by  $\mathbb{E}[L(1, \gamma(A, B))]$ . If, in addition,  $\gamma$  is deterministic then we have

$$\int_{\gamma(A,B)} dB_z \in \mathcal{N}(0, L(1, \gamma(A, B))),$$

which shows that the line stochastic integral of the first kind is essentially path dependent.

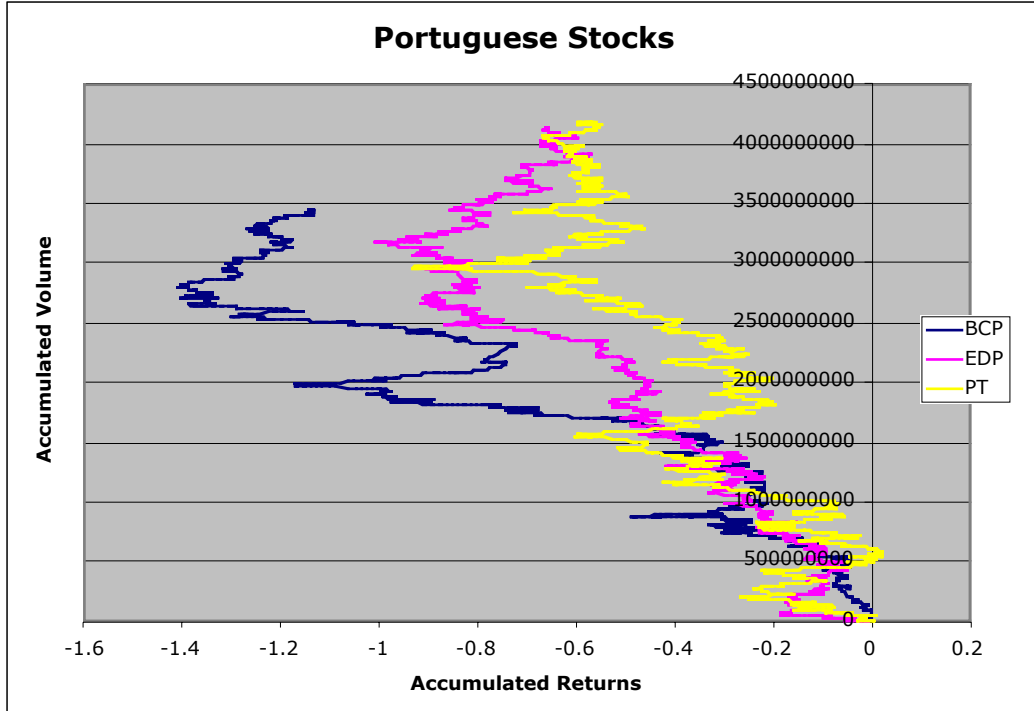
**3. A model for risk comparison.** In this section we will present a comparison of different stocks under the perspective of a risk measure having as factors price and liquidity given by volume of transactions. This example is presented for illustration of the method in order to show that some interesting results may be derived even this very simple framework. We suppose that for each stock price and volume are given by:

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dB_t^{(1)} & S_0 = S(0) \\ dV_t = \nu V_t dt + V_t dB_t^{(2)} & V_0 = V(0) \end{cases} \quad (20)$$

with  $\mu, \sigma, \nu, \rho, V(0), S(0)$  constants. As for the correlation structure of the brownian processes  $(B_t^{(1)})_{t \geq 0}$  and  $(B_t^{(2)})_{t \geq 0}$  we performed some statistical tests on the residues of the adjusted series with results implying that we may safely suppose that these brownian processes are independent. We associate to each stock the line in the plane

$$\gamma_t = \left( \int_0^t r_s ds, \int_0^t V_s ds \right) \quad (21)$$

where  $(r_s)_{s \geq 0}$  is the daily return of the stock. A plot of this line for each of the stocks considered is presented in the next figure.



We will consider the risk as defined, for each stock by:

$$R_{z(t)} = \int_{\gamma(A,z(t))} dB_w \quad (22)$$

where  $z(t)$  is the running point of  $\gamma^*$  and  $(B_t)_{t \geq 0}$  is a brownian process independent of the previous ones. This very simplified risk function amounts to postulate an uniform risk over the plane defined by the variables price and volume of transactions. In order to compare the stocks under this risk we will estimate the tail of the maximum risk during a period of time  $[0, t]$  using Doob's inequality.

**Proposition 3.1.** *For the risk defined by formula 22 we have for  $\lambda > 0$  and  $t > 0$*

$$\mathbb{P} \left[ \sup_{w \in \gamma(A,z(t))} |R_w| \geq \lambda \right] \leq \frac{1}{\lambda^2} \int_0^t \left( \left( \mu - \frac{\sigma^2}{2} \right)^2 + \sigma^2 + V_0 e^{(2\nu + \rho^2)u} \right)^{1/2} du. \quad (23)$$

**Proof.** By Doob's inequality 7 we will have to estimate  $\mathbb{E}[|R_{z(t)}|^2]$ . As we have

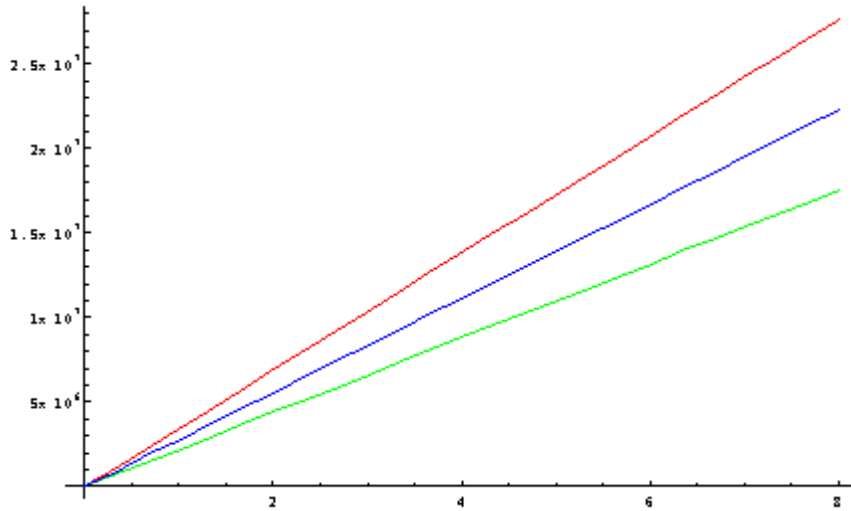
$$\begin{aligned} \mathbb{E}[|R_{z(t)}|^2] &= \mathbb{E} \left[ \left| \int_{\gamma(A,z(t))} dB_w \right|^2 \right] = \mathbb{E} \left[ \left| \int_0^t \sqrt{\gamma'(s)} dB_s \right|^2 \right] \stackrel{(a)}{\leq} \\ &\leq \mathbb{E} \left[ \int_0^t |\gamma'(t)| ds \right] = \mathbb{E} \left[ \int_0^t \sqrt{r_s^2 + V_s^2} ds \right] \stackrel{(b)}{\leq} \int_0^t \sqrt{\mathbb{E}[r_s^2] + \mathbb{E}[V_s^2]} ds \stackrel{(c)}{=} \\ &= \frac{1}{\lambda^2} \int_0^t \left( \left( \mu - \frac{\sigma^2}{2} \right)^2 + \sigma^2 + V_0 e^{(2\nu + \rho^2)u} \right)^{1/2} du \end{aligned}$$

because

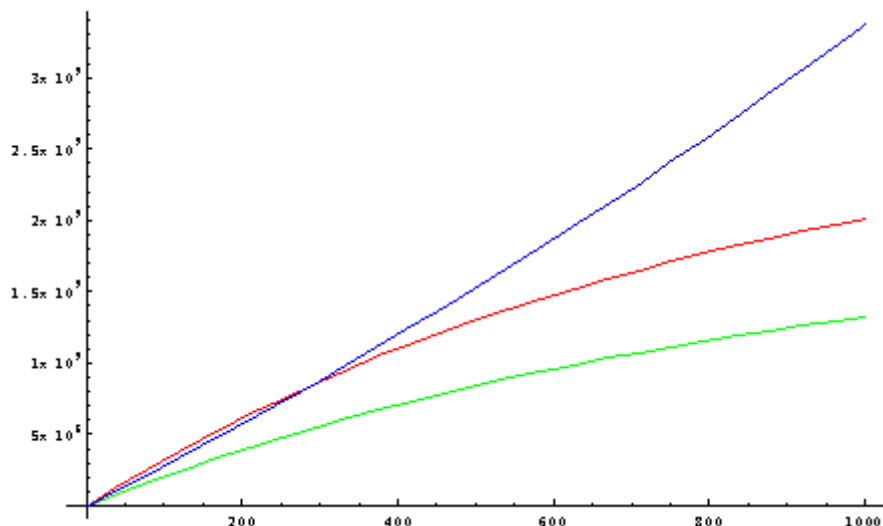
- (a) Ito's inequality,
- (b) Fubini theorem and Jensen inequality,
- (c) simple calculations with the integrated form of equations 20,

the proof is complete. ■

In order to compare the stocks we present in the next figures the plots of the function of the variable  $t$  defined for each stock by the right term of formula 23.



The line in red (corresponding to the greatest values) is obtained from BCP (Banco Comercial Português) data. The line in green (corresponding to the smallest values) is obtained from EDP (Electricidade de Portugal) and the intermediate line corresponds to PT (Portugal Telecom) data. It is visible that the three stocks are clearly distinct under this risk measure with the differences being amplified with time. The following figure is similar to the first one but it covers an interval of 2.5 years instead of eight days.



We observe that around 300 days there is a change on the relative risk profile of the stocks.

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