ON THE APPROXIMATION OF THE SOLUTIONS OF THE RIEMANN PROBLEM FOR A DISCONTINUOUS CONSERVATION LAW

JOÃO-PAULO DIAS

CMAF/UL, Av. Prof. Gama Pinto, 2, 1649-003 Lisboa-Portugal
dias@ ptmat.fc.ul.pt

MÁRIO FIGUEIRA

CMAF/UL, Av. Prof. Gama Pinto, 2, 1649-003 Lisboa-Portugal
figueira@ ptmat.fc.ul.pt

Abstract. For a class of discontinous flux functions introduced in [3] (cf. also [4]), we prove, for the Riemann problem, an extension of the existence result proved in [2] for a Lipschitz continuous flux function. In the last section, and based in the previous results, we apply the Lax-Friedrichs approximation method and the limiters technique (cf.[5]) to compute the quoted solution in a numerical example.

Keywords: Phase transitions; discontinous conservation laws; numerical approximation.

1. Introduction and main results.

In the study of phase transitions (cf. [7], [9]) one can consider some limit cases that corresponds to a discontinuous flux function. In [3] it has been introduced an appropriate notion of entropy weak solution to the Cauchy problem for the corresponding conservation law and an existence theorem has been proved (cf. also [4] for related results). Let us consider the flux function (discontinuous at the origin) defined by

$$f(u) = g(u) + (h(u) - g(u))H(u) \tag{1.1}$$

where $g, h \in C^\infty(\mathbf{R})$ verify $g(0) = 1$, $h(0) = 0$, and H is the Heaviside function (defined by $H(u) = 1$ if $u > 0$ and $H(u) = 0$ if $u < 0$). We recall the following two definitions (cf.[3]), where $\tilde{f}$ denotes the the multivaluated function defined by

$$\tilde{f}(u) = f(u) \text{ if } u \neq 0, \quad \tilde{f}(0) = [0,1] :$$

Definition 1.1. *A function $u \in L^\infty(\mathbf{R}\times]0,+\infty[)$ is called a weak solution to the Cauchy problem for the equation*

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \tag{1.2}$$

with initial data $u_0 \in L^\infty(\mathbf{R})$, if there exists a function $v \in L^\infty(\mathbf{R}\times]0,+\infty[)$ such that $v(x,t) \in \tilde{f}(u(x,t))$ a.e. and

$$\int_0^{+\infty}\int_{\mathbf{R}} u\frac{\partial \varphi}{\partial t} dxdt \; + \int_0^{+\infty}\int_{\mathbf{R}} v\frac{\partial \varphi}{\partial x} dxdt \; + \int_{\mathbf{R}} u_0(x)\varphi(x,0)dx = 0$$

for each $\varphi \in C^1_c(\mathbf{R} \times [0,+\infty[)$ (where $\varphi \in C^1_c$ means $\varphi \in C^1$ with compact support).

Throughout the paper, $\tilde{H}$ will be the multivaluated function defined by $\tilde{H}(u) = H(u)$ if $u \neq 0$, $\tilde{H}(0) = [0,1]$.

Definition 1.2. *A weak solution u of the Cauchy problem for the equation (1.2) is called an entropy weak solution if, for each entropy $\eta \in C^1(\mathbf{R})$, η convex, there exists a function $w \in L^\infty(\mathbf{R}\times]0,+\infty[)$ such that $w(x,t) \in \tilde{H}(u(x,t))$ a.e. and*

$$\frac{\partial}{\partial t}\eta(u) + \frac{\partial}{\partial x}F(u) - \eta'(0)\frac{\partial w}{\partial x} \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbf{R}\times]0,+\infty[) \tag{1.3}$$

where

$$F(u) = \int_0^u \eta'(s)[g'(s) + (h'(s) - g'(s))H(s)]ds.$$

The following notations and functions will be used throughout this paper:

(*i*)

$$f_\varepsilon(u) = g(u) + (h(u) - g(u)) \int_{-\varepsilon}^{u} \rho_\varepsilon(s)ds, \quad u \in \mathbf{R},$$

where $\rho_\varepsilon(s) = \frac{1}{\varepsilon}\rho\left(\frac{s}{\varepsilon}\right)$, $\varepsilon > 0$, $\rho \geq 0$, $\rho \in \mathcal{D}(\mathbf{R}) = C_c^\infty(\mathbf{R})$ with support $[-1, 1]$ and such that $\rho(-s) = \rho(s)$ and $\int_{\mathbf{R}} \rho(s)ds = 1$ (mollifiers);

(*ii*) u_L and u_R with $u_L < 0 < u_R$ are the Riemann data (real constants);

(*iii*) $\bar{f}$ (respectively $\bar{f}_\varepsilon$) denotes the lower convex envelope of f (with $f(0) = 0$) (respectively of f_ε) in the interval $[u_L, u_R]$.

It is easy to see that $\bar{f}$ is a Lipschitz function in $[u_L, u_R]$, $\bar{f}'$ having at most a discontinuity at the origin, and that $-\infty < \bar{f}'(u_L) < \bar{f}'(u_R) < +\infty$. Moreover f_ε is smooth and $f_\varepsilon(u) \underset{\varepsilon \to 0}{\longrightarrow} f(u)$, $u \neq 0$ ($f_\varepsilon(0) = 1/2$, $\varepsilon > 0$). We will prove the following lemma:

Lemma 1.3. *The sequence* $\{\bar{f}_\varepsilon\}_{\varepsilon > 0}$ *is bounded in* $W^{1,\infty}(I)$, $I =]u_L, u_R[$, *for* $\varepsilon \leq \varepsilon_0$. *There exists a subsequence, still denoted by* $\{\bar{f}_\varepsilon\}_{\varepsilon > 0}$, *such that* $\bar{f}_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \bar{f}$ *in* $C(\bar{I})$.

Now, for the initial data (Riemann data)

$$u_0(x) = \begin{cases} u_L & \text{if} \quad x < 0 \\ u_R & \text{if} \quad x > 0 \end{cases} \tag{1.4}$$

we can consider (cf.[2]) the Cauchy problem for the equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\bar{f}(u) = 0. \tag{1.5}$$

In [2] the authors constructed the unique bounded entropy weak solution u (in the Kruzkov's sense, verifying the usual continuity property at $t = 0$ (cf.[6])) for the Cauchy problem (1.5), (1.4), which is a self-similar function (in the same sense) of the Riemann problem

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + \dfrac{\partial}{\partial x} f_\varepsilon(u_\varepsilon) = 0 \\ u_\varepsilon(x, 0) = u_0(x), \quad u_0 \text{ defined by (1.4)} \end{cases} \tag{1.6}$$

It is well known (cf.[8]) that in (1.6) we can replace f_ε by $\bar{f}_\varepsilon$ to obtain the same solution u_ε. By applying lemma 1.3 and theorem 3.1, (iii), of [1] to the functions $\bar{f}_\varepsilon$ we will prove the following theorem:

Theorem 1.4. *Let* $\{u_\varepsilon\}_{\varepsilon>0}$ *be the sequence of (self-similar) functions solutions of* (1.6) *for each* $\varepsilon > 0$*. Then, there is a subsequence still denoted by* $\{u_\varepsilon\}_{\varepsilon>0}$*, such that* $u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u$ *in* $L^\infty(\mathbf{R}\times]0,+\infty[)$ *weak* $*$ *and a.e. in* $\mathbf{R}\times]0,+\infty[$*, where* u *is the unique bounded entropy weak solution (in the Kruzkov's sense) of* (1.5), (1.4)*. Moreover,* u *is an entropy weak solution (in the sense of definition* 1.1 *and* 1.2*) of the Cauchy problem* (1.2), (1.4).

In the last section of this paper we deal with the numerical approximation of the solution of the quoted Riemann problem for the equation (1.2), obtained by application of theorem 1.4. For that, and since in general it is difficult to construct the lower convex envelope $\bar{f}$ of the original flux function f, we introduce a procedure to obtain a new Lipschitz continuous flux function f^M with the same lower convex envelope $\bar{f}$ in $[u_L, u_R]$. By theorem 3.3 in [2] we obtain the same solution u for the corresponding Riemann problem. Then, we apply the Lax-Friedrichs method and the limiters technique (cf.[5]) to compute the quoted solution in an example.

2. Proof of theorem 1.4.

We start with the proof of lemma 1.3. With the notations introduced in §1 and putting $f(0) = f(0^+) = h(0) = 0$ ($f(0^-) = g(0) = 0$) it is easy to see that $f_\varepsilon(u) = f(u)$ for $|u| \geq \varepsilon$, $f_\varepsilon(0) = \frac{1}{2}$ and $\bar{f}(0) \leq 0$. By continuity, we derive

$$\bar{f}(u) \leq f_\varepsilon(u), \quad u \in [u_L, u_R] \quad \text{for} \quad \varepsilon \leq \varepsilon_0.$$

Hence,

$$\bar{f}(u) \leq \bar{f}_\varepsilon(u), \quad u \in [u_L, u_R] \quad \text{for} \quad \varepsilon \leq \varepsilon_0. \tag{2.1}$$

On the other hand, there exists $c > 0$ such that

$$\bar{f}_\varepsilon(u) \leq f_\varepsilon(u) \leq c, \quad u \in [u_L, u_R]. \tag{2.2}$$

We derive, with $I =]u_L, u_R[$,

$$\|\bar{f}_\varepsilon\|_{L^\infty(I)} \leq c_1, \quad \text{for} \quad \varepsilon \leq \varepsilon_0.$$

Moreover we have, by convexity,

$$-\infty < \bar{f}'(u_L) \leq \bar{f}'_\varepsilon(u_L) \leq \bar{f}'_\varepsilon(u) \leq \bar{f}'_\varepsilon(u_R) \leq \bar{f}'(u_R) < +\infty,$$

for $u \in \bar{I}$, $\varepsilon \leq \varepsilon_0$. Hence,

$$\|\bar{f}_\varepsilon\|_{W^{1,\infty}(I)} \leq c_2, \quad \text{for } \varepsilon \leq \varepsilon_0.$$

Since the injection $W^{1,\infty}(I) \hookrightarrow C(\bar{I})$ is compact, we can extract a subsequence of $\{f_\varepsilon\}_{\varepsilon>0}$, still denoted by $\{f_\varepsilon\}_{\varepsilon>0}$, such that

$$\bar{f}_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \theta \quad \text{in} \quad C(\bar{I}).$$

The function θ is convex and, by (2.1),

$$\bar{f}(u) \leq \theta(u), \quad u \in \bar{I}.$$

By (2.2) we derive $\theta(u) \leq f(u)$ for $u \in \bar{I}$, $u \neq 0$, and so, by right continuity at the origin,

$$\theta(u) \leq f(u) \quad \text{for} \quad u \in \bar{I}.$$

Hence,

$$\theta(u) \leq \bar{f}(u) \quad \text{for} \quad u \in \bar{I}$$

and the lemma is proved. ∎

We are now able to prove theorem 1.4. To do this we consider, for $\varepsilon, \delta \leq \varepsilon_0$ in lemma 1.3, the Riemann problem (1.6) with u_0 defined by (1.4), that is, equivalently, the Riemann problems

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + \dfrac{\partial}{\partial x}\bar{f}_\varepsilon(u_\varepsilon) = 0 \\ u_\varepsilon(x,0) = u_0(x) \end{cases} \qquad \text{and} \qquad \begin{cases} \dfrac{\partial u_\delta}{\partial t} + \dfrac{\partial}{\partial x}\bar{f}_\delta(u_\delta) = 0 \\ u_\delta(x,0) = u_0(x) \end{cases}$$

and we apply the estimate (iii) in theorem (3.1) of [1]. We deduce, for each $R > 0, t \geq 0$,

$$\int_{|x|\leq R} |u_\varepsilon(x,t) - u_\delta(x,t)|\, dx \leq$$
$$\leq c_0 \left[(R + Mt)\,(u_R - u_L)\, t\, \|(\bar{f}_\varepsilon - \bar{f}_\delta) - (\bar{f}_\varepsilon - \bar{f}_\delta)(0)\|_{L^\infty(I)} \right]^{1/2}$$

with $M = \sup\limits_{\varepsilon\leq\varepsilon_0} \|\bar{f}_\varepsilon'\|_{L^\infty(I)} < +\infty$ by lemma 1.3 and since $u_L \leq u_\varepsilon(x,t) \leq u_R$ a.e. in $\mathbf{R}\times]0,+\infty[$. From the previous estimates and lemma 1.3, we easily derive that there exists a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$, still denoted by $\{u_\varepsilon\}_{\varepsilon>0}$, and a function u in $L^\infty(\mathbf{R}\times]0,+\infty[)$ such that

$$u_L \leq u(x,t) \leq u_R \quad \text{a.e. in} \quad \mathbf{R}\times]0,+\infty[,$$

$$u_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u \quad \text{in} \quad L^\infty(\mathbf{R}\times]0,+\infty[) \text{ weak}*, \quad \text{in} \quad L^1_{loc}(\mathbf{R}\times[0,+\infty[)$$

and a.e. in $\mathbf{R}\times]0,+\infty[$.

Moreover, since

$$\int_{|x|\leq R} |u(x,t)-u_0(x)|\,dx \leq \int_{|x|\leq R} |u(x,t)-u_\varepsilon(x,t)|\,dx + \\ + \int_{|x|\leq R} |u_\varepsilon(x,t)-u_0(x)|\,dx,$$

one can easily prove that, for $t \notin A \subset [0,+\infty[$, A with zero mesure,

$$\lim_{t\to 0, t\notin A} \int_{|x|\leq R} |u(x,t)-u_0(x)|\,dx = 0, \quad \forall R > 0.$$

Hence, by [6], [2] and lemma 1.3, u is the unique bounded entropy weak solution (in the Kruzkov's sense) of the Riemann problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\bar{f}(u) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

and u is a self-similar function.

Now recall that u_ε is also the unique bounded entropy weak solution of the Riemann problem (1.6) and that we can assume, by lemma 1 in [3] that

$$\int_{-\varepsilon}^{u_\varepsilon} \rho_\varepsilon(s)\,ds \underset{\varepsilon\to 0}{\rightharpoonup} w \quad \text{in} \quad L^\infty(\mathbf{R}\times]0,+\infty[) \text{ weak}*$$

with $w \in \tilde{H}(u)$. We derive for $f_\varepsilon(u_\varepsilon)$,

$$f_\varepsilon(u_\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} v = g(u) + (h(u)-g(u))\,w \quad \text{in} \quad L^\infty(\mathbf{R}\times]0,+\infty[) \text{ weak}*$$

and

$$v(x,t) \in \tilde{f}(u(x,t)) \quad \text{a.e.} \quad \mathbf{R}\times]0,+\infty[,$$

where $\tilde{f}$ is the multivalued function defined in §1. Hence, by passing to the limit in (1.6), we obtain that u is a weak solution of the Riemann problem

$(1.2), (1.4)$ in the sense of definition 1.1. Finally, let $\eta \in C^1$ be a convex entropy and let

$$F_\varepsilon(s) = \int_{-\varepsilon}^{s} \eta'(y) f'_\varepsilon(y) dy, \quad s \in \mathbf{R},$$

be the corresponding entropy flux for the approximate problem (1.6). We have

$$\frac{\partial}{\partial t}\eta(u_\varepsilon) + \frac{\partial}{\partial x}F_\varepsilon(u_\varepsilon) \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbf{R}\times]0, +\infty[).$$

It is easy to see (cf.[3], proof of theorem 1) that

$$F_\varepsilon(u_\varepsilon) \underset{\varepsilon \to 0}{\rightharpoonup} F(u) - \eta'(0)w \quad \text{in} \quad L^\infty(\mathbf{R}\times]0, +\infty[) \text{ weak}*,$$

where

$$F(u) = \int_0^u \eta'(s)[g'(s) + (h'(s) - g'(s))H(s)]ds.$$

Hence, we derive

$$\frac{\partial}{\partial t}\eta(u) + \frac{\partial}{\partial x}F(u) - \eta'(0)\frac{\partial w}{\partial x} \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbf{R}\times]0, +\infty[)$$

and it follows that u is an entropy weak solution in the sense of definition 1.2 and the theorem 1.4 is proved. ■

3. Numerical examples.

With the usual notations (cf.[5]) we represent by Δt a uniform time step and by Δx the increment space, and we put

$$r = \frac{\Delta t}{\Delta x}.$$

For the sake of simplicity, we consider the explicit difference scheme of 3--points in conservative form,

$$v_j^{n+1} = \mathcal{H}(v_{j-1}^n, v_j^n, v_{j+1}^n) = v_j^n - r\,[\,g(v_{j-1}^n, v_j^n) - g(v_j^n, v_{j+1}^n)\,], \qquad j \in \mathbf{Z},\ n = 0, 1, \ldots, \tag{3.1}$$

where $g : \mathbf{R}^2 \longrightarrow \mathbf{R}$, the numerical flux, depends on the flux function f (recall that a difference scheme (3.1) is said consistent with the equation (1.2) if $g(v, v) = f(v),\ \forall v \in \mathbf{R}$).

Recall now that the theory of numerical approximation for scalar conservation laws uses the lipschitzian condition for g as a fundamental assumption in the theorems of convergence and so we cannot apply directly the schemes in our case. If we take, for instance, the "very stable" Lax--Friedrichs first order scheme,

$$v_j^{n+1} = v_j^n - r\,[\,g_{j+1/2}^n - g_{j-1/2}^n\,],$$

where

$$g_{j+1/2} = g^{LF}(v_j^n, v_{j+1}^n),$$

$$g^{LF}(v,w) = \frac{f(v)+f(w)}{2} - \frac{w-v}{2r},$$

and if we apply to the simple case,

$$f(x) = \begin{cases} x+1 & x<0 \\ x & x \geq 0 \end{cases}, \qquad u_L = 0.5,\ u_R = 1,$$

we get the catastrophic approximation to the solution $u(x,t)$ at $t = 0.5$,

$$u(x,0.5) = \begin{cases} -0.5 & \text{if} \quad x < -0.5 \\ 0 & \text{if} \quad -0.5 < x < 0.5 \\ 1 & \text{if} \quad x > 0.5 \end{cases}$$

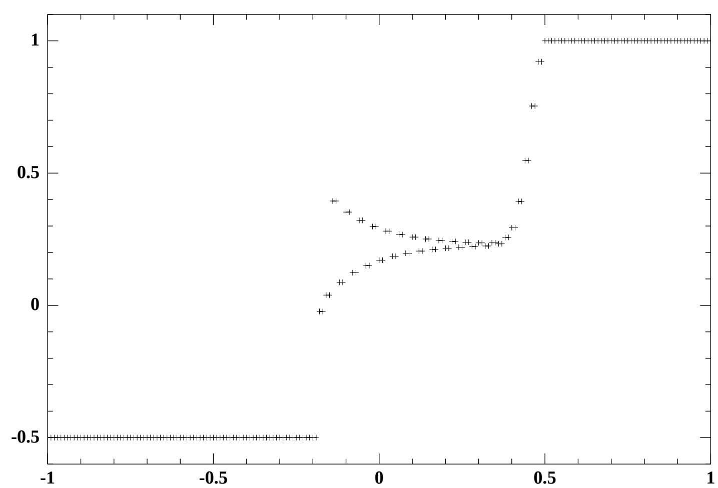

Fig. 1.

We can observe a clear difficult in convergence near $u(x,0.5) = 0$, precisely the point of discontinuity of the flux function f.

However, using the results obtained in the previous sections, one can apply the numerical schemes to the lower convex envelope of f, $\bar{f}$, in the interval $[u_L, u_R]$, which is a lipschitzian function. Besides, one can even take any lipschitzian function, say f^M, such that the lower convex envelope in the interval $[u_L, u_R]$ is equal to $\bar{f}$. In fact, using theorem 3.3 in [2] we get always the entropy weak solution of our problem. Now, remark that if we take $-\delta < 0$, sufficiently close to 0, the modified flux f^M defined by

$$\begin{cases} f^M(x) = kx, \;\; k = -\frac{f(-\delta)}{\delta}, & \text{if} \quad -\delta \le x < 0 \\ f^M(x) = f(x) \quad \text{elsewhere} & \end{cases}$$

certainly admits the same lower convex envelope of f. Nevertheless, if we use this modified flux in the numerical approximation we get a very severe CFL condition,

$$r. \sup_{x \in [u_L, u_R]} |f'(x)| \le 1,$$

and so $\frac{\Delta t}{\Delta x}.k \le 1$, which penalize the eficiency of the scheme. Instead, we propose the following procedure: let $(\bar{u}, f(\bar{u}))$ the first intersection point of the graph of f with the segment connecting the origin to the point $(u_L, f(u_L))$. We define now:

$$f^M(x) = \begin{cases} f(x) & \text{if} \quad x \le \bar{u} \\ kx & \text{if} \quad \bar{u} < x < 0 \\ f(x) & \text{if} \quad x \ge 0 \end{cases} \tag{3.2}$$

where, $k = f(\bar{u})/\bar{u}$. It is clear that the lower convex envelope of f^M in the interval $[u_L, u_R]$ is $\bar{f}$ and we can now apply the usual TVD schemes consistents with any entropy condition. We illustrate the numerical approximation with the following example:

Consider the discontinous flux function f defined by

$$f(x) = \begin{cases} -\frac{3}{2}x - 1 & \text{if} \quad x \le -4/5 \\ x + 1 & \text{if} \quad -4/5 < x < 0 \\ x/2 & \text{if} \quad x \ge 0 \end{cases}$$

and represented in Fig.2

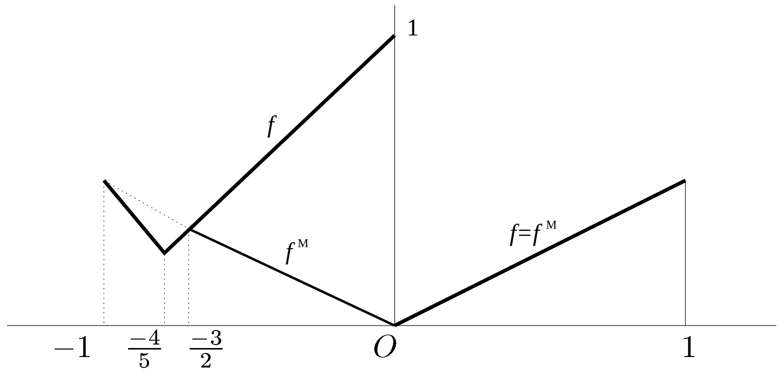

Fig. 2.

The entropy solution coresponding to the Riemann data, $u_L = -1$, $u_R = 1$, is given by

$$u(x,t) = \begin{cases} -1 & \text{if} \quad x/t \leq -3/2 \\ -4/5 & \text{if} \quad -3/2 < x/t \leq -1/4 \\ 0 & \text{if} \quad -1/4 < x/t \leq 1/2 \\ 1 & \text{if} \quad x/t > 1/2 \end{cases} \tag{3.3}$$

Accordingly with the procedure (3.2), we define $f^M(x) = -x/2$ if $-2/3 \leq x < 0$ and $f^M(x) = f(x)$ elsewhere. Now, we use a limiter technique to approximate the solution (3.3) (cf.[5], pag.187).

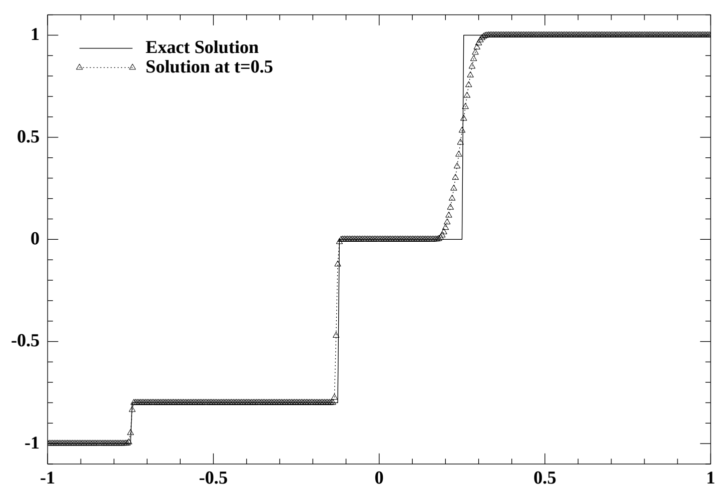

Fig. 3.

More precisely, we take as the underlying entropic scheme a variant of the Lax-Friedrichs scheme,

$$v_j^{n+1} = \frac{v_{j+1}^n + 2v_j^n + v_{j-1}^n}{4} - r\,\frac{f_{j+1}^n - f_{j-1}^n}{2},$$

and we choose the superbee limiter,

$$\varphi(z) = \max\{0, \min\{2z, 1\}, \min\{z, 2\}\}.$$

The approximate solution at $t = 0.5$, with a stepsize $\Delta x = 0.005$, is displayed in Fig. 3.

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