The effect of inversely unstable solutions on the attractor of the forced pendulum equation with friction

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Abstract: Consider the attractor $\mathcal{A}$ of a periodically forced equation of pendulum type with linear friction, in the cylinder. M. Levi and independently Min, Xian, and Jinyan show that if the friction coefficient is larger than a certain bound then $\mathcal{A}$ is homeomorphic to the circle. We shall give a topological version of the definition of inversely unstable solution of N. Levinson and show that the appearance of such solutions imply that $\mathcal{A}$ is not homeomorphic to the circle. As an application we shall show that the bounds on the friction coefficient obtained before are optimal.

Keywords: Attractor, pendulum, inversely unstable solution, rotation number.

1 Introduction

The purpose of this paper is the study of the dynamical behaviour of a forced oscillator of pendulum type with friction

$$x'' + h(x)x' + g(t,x) = 0 \tag{1}$$

where h and g are smooth functions, 1-periodic on x and T-periodic on t. Moreover we will suppose that

$$0 < \min_{\mathbb{R}} h(x) = c.$$

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We can write equation (1) as

$$\begin{cases} y_1' = y_2 - H(y_1) \\ y_2' = -g(t, y_1) \end{cases}, \tag{2}$$

where $H(x) = \int_0^x h(s)ds$. Thus x is a solution of (1) iff $(x, x' + H(x))$ is solution of (2). It is not difficult to prove that the solutions of (2) are globally defined and are unique for each set of initial conditions. Consider the Poincaré map $P : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $P(y_0) = y(T; 0, y_0)$, where $y(t) = y(t; 0, y_0)$ is the solution of (2) such that $y(0) = y_0$. Notice that the vector field $(y_1, y_2) \to (y_2 - H(y_1), -g(t, y_1))$ is periodic with period $R = (1, \overline{h})$, where $\overline{h} = \int_0^1 h(s)ds$. This implies that

$$P(y_0 + R) = P(y_0) + R. \tag{3}$$

Moreover, if y is a solution of (2) then $y + kR,\ k \in \mathbb{Z}$ is also a solution; we shall regard them as the same solution. Consider the equivalence relation $\sim$ in $\mathbb{R}^2$ where two elements are in relation iff $x = y + kR,\ k \in \mathbb{Z}$. The quotient space $\mathcal{C} = \mathbb{R}^2 / \sim$ is topologically the cylinder and we will consider the induced metric on it. The elements of C will be denoted by $\overline{x}$ where $x \in \mathbb{R}^2$.

The Poincaré map is well defined on $\mathcal{C}$ by $\overline{x_0} \to \overline{P(x_0)}$. Defining

$$\overline{B}_\rho = \{\overline{x} \in \mathcal{C} / | < x, (\overline{h}, -1) > | \leq \rho\},$$

it is not difficult to prove that we can fix ρ_0 sufficiently large in such a way that $\overline{P}(\overline{B}_{\rho_0}) \subset \overline{B}_{\rho_0}$ and for each $\overline{x}$ there exists $n \in \mathbb{N}$ such that $\overline{P}^n(\overline{x}) \in \overline{B}_{\rho_0}$. This property illustrates the dissipative nature of problem (2). Consider the sequence of sets

$$\overline{B}_{\rho_0} \supset \overline{P}(\overline{B}_{\rho_0}) \supset \overline{P}^2(\overline{B}_{\rho_0}) \ldots$$

and define $\mathcal{A} = \bigcap_{n \in \mathbb{N}} \overline{P}^n(\overline{B}_{\rho_0})$, which is a non-empty, compact, and connected set. Moreover, the definition of $\mathcal{A}$ does not depend of ρ_0, $\mathcal{A}$ is an invariant set for $\overline{P}$, and a global attractor of the orbits given by iterates of the Poincaré map (see [4]). It can be proved that the Čech cohomology of $\mathcal{A}$ and the cylinder are isomorphic (see [12]) and consequently $\mathcal{A}$ is not contractible in the cylinder. M. Levi [5], Q. Min, S. Xian and Z. Jinyan [7], dealing with some particular forms of h and g, proved that $\mathcal{A}$ is homeomorphic to the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, if the damping coefficient is large enough. In this paper we shall give conditions under which $\mathcal{A}$ is not homeomorphic to $\mathbb{T}^1$.

Given $(a, b) \in \mathbb{Z} \times \mathbb{N},\ b \geq 1$, we shall say that a solution y of (2) is (a, b)-periodic iff

$$y(t + bT) = y(t) + aR.$$

Those solutions correspond to bT-periodic solutions on $\mathcal{C}$ which winds around the cylinder a times before closing. If y is an (a,b)-periodic solution of (2) then $y(0)$ is a fixed point of $P^b - aR$. If it is an isolated fixed point then the index of y can be defined as

$$\gamma_b(y) = deg(I - [P^b - aR], B),$$

where B is a small ball in $\mathbb{R}^2$ such that $y(0)$ is the only fixed point of $P^b - aR$ on B. Such a solution is also $(2a, 2b)$-periodic and if $y(0)$ is an isolated fixed point of $P^{2b} - 2aR$ then we can consider the correspondent index $\gamma_{2b}(y)$. Given an (a,b)-periodic solution of (2) such that $y(0)$ is an isolated fixed point of $P^b - aR$ and $P^{2b} - 2aR$, we shall say that y is inversely unstable iff

$$\gamma_b(y) = 1 \text{ and } \gamma_{2b}(y) = -1.$$

The main result of this paper is the following:

Theorem 1 *Suppose that for some $(a,b) \in \mathbb{Z} \times \mathbb{N}$, $b \geq 1$, the set of (a,b)-periodic solutions of (2) is finite. If there exists an inversely unstable (a,b)-periodic solution then $\mathcal{A}$ is not homeomorphic to $\mathbb{T}^1$.*

Given an $(a,b)-$periodic solution $y = (y_1, y_2)$ of (2) we consider the linearized equation

$$\begin{cases} \xi' = \eta - h(y_1(t))\xi \\ \eta' = -\frac{\partial g}{\partial x}(t, y_1(t))\xi \end{cases}, \tag{4}$$

that is bT-periodic in t. The function $Y(t) = \frac{\partial}{\partial y_0} y(t; 0, y_0)$ is the solution of (4) such that $Y(0) = I_2$. The eigenvalues of $Y(bT)$ will be called the characteristic multipliers of (4). Notice that by Jacobi-Liouville's formula $0 < \mu_1\mu_2 \leq e^{-cbT}$. In the case that $\mu_1 < -1 < \mu_2 < 0$ we shall say that the linearized equation is inversely unstable. If one of the characteristic multipliers has the value 1, then we shall say that y is degenerate. On the other hand $P'(y(0)) = Y(bT)$ so if $y(0)$ is an isolated fixed point of $P^b - aR$ and y is not degenerate then

$$\gamma_b(y) = sign\{(1 - \mu_1)(1 - \mu_2)\}.$$

Levinson [6] defined an inversely unstable solution to be a solution such that the characteristic multipliers of the linearized equation μ_1, μ_2 satisfy

$$\mu_1 < -1 < \mu_2 < 0.$$

If y is an (a,b)-periodic, inversely unstable solution in the sense of Levinson then $y(0)$ is an isolated fixed point of $P^b - aR$ and $P^{2b} - 2aR$. Hence we have

$$\gamma_b(y) = sign\{(1 - \mu_1)(1 - \mu_2)\} = 1$$

and

$$\gamma_{2b}(y) = sign\{(1-\mu_1^2)(1-\mu_2^2)\} = -1.$$

Thus y is inversely unstable under our definition. On the other hand, if $\mu_1 = -1 < \mu_2 < 0$ it is possible to have an inversely unstable solution under our definition and not under Levinson's definition. We observe that our definition of inverse instability is of different nature. Actually our definition only depends on the degree, a topological invariant.

In section 3 we shall construct an example to which we apply the last Theorem. In particular this will show that some results obtained in [7] are optimal. More precisely, we shall prove that:

Theorem 2 *Given $\mathcal{H} > c^2/4$ there exists $g \in C^\infty(\mathbb{R}/\mathbb{Z})$, $k \in \mathbb{N}$, and $p \in C(\mathbb{R}/kT\mathbb{Z})$ such that $g' < \mathcal{H}$ and the equation*

$$\begin{cases} x_1' = x_2 - cx_1 \\ x_2' = -g(x_1) + p(t) \end{cases} \tag{5}$$

has a finite number of $(0,k)-$periodic solutions and one of them is inversely unstable. Hence the attractor is not homeomorphic to $\mathbb{T}^1$.

This paper is based on the PhD program of R. Martins supervised by Professor R. Ortega, the interested reader could find related results in the draft of the PhD thesis in http://ptmat.lmc.fc.ul.pt/~rmartins

2 Inversely unstable solutions imply $\mathcal{A} \not\cong \mathbb{T}^1$

Let $C^1(\mathbb{R}/\mathbb{Z})$ be the set of 1-periodic functions in $C^1(\mathbb{R})$ and $C^{0,1}(\mathbb{R}/T\mathbb{Z} \times \mathbb{R}/\mathbb{Z})$ be the set of functions in $C^0(\mathbb{R}^2)$, T-periodic on the first variable, 1-periodic on the second, and such that the function $(t,x) \to \frac{\partial g}{\partial x}(t,x)$ is well defined and continuous in $\mathbb{R}^2$. If $h \in C^1(\mathbb{R}/\mathbb{Z})$ and $g \in C^{0,1}(\mathbb{R}/T\mathbb{Z} \times \mathbb{R}/\mathbb{Z})$ then the Poincaré map is well defined on $\mathbb{R}^2$ and of class C^1. Let $\mathcal{A} \subset \mathcal{C}$ be the invariant set for $\overline{P}$, defined in the introduction. When $\mathcal{A}$ is homeomorphic to $\mathbb{T}^1$ the dynamics on the phase space is understood. Indeed, if $\phi : \mathcal{A} \to \mathbb{T}^1$ is an homeomorphism and $\tau : \mathbb{R} \to \mathbb{T}^1$ the canonical projection then we can take a lift of $\overline{P}_{/\mathcal{A}}$ to the real line, i.e. a function $f : \mathbb{R} \to \mathbb{R}$ such that $\overline{P}_{/\mathcal{A}} \circ \phi^{-1} \circ \tau = \phi^{-1} \circ \tau \circ f$. Moreover, f should satisfy

$$f(t+1) = f(t) + 1 \text{ or } f(t+1) = f(t) - 1.$$

We shall say that $\overline{P}_{/\mathcal{A}}$ is orientation-preserving iff the first situation occurs.

Intuitively, the Poincaré map is an orientation-preserving homeomorphism in the cylinder, although it is not obvious that $\overline{P}_{/\mathcal{A}}$ is orientation-preserving; see for example that the application $\overline{(\theta, r)} \to \overline{(-\theta, -r/2)}$ preserves the orientation on the cylinder but it does not preserves the orientation on the invariant curve $r = 0$. We shall prove in a rigorous way that $\overline{P}_{/\mathcal{A}}$ is orientation-preserving. The following Lemma has an elementary proof.

Lemma 3 *If $\mathcal{A}$ is homeomorphic to $\mathbb{T}^1$ and $\overline{P}_{/\mathcal{A}}$ it is not orientation-preserving then there exists precisely two fixed points of $\overline{P}_{/\mathcal{A}}$, say A and B, that divides $\mathcal{A}$ in two arcs C_+ and C_- ($\mathcal{A} = A \dot{\cup} B \dot{\cup} C_+ \dot{\cup} C_-$) such that $\overline{P}(C_+) = C_-$ and $\overline{P}(C_-) = C_+$.*

Proposition 4 *If $\mathcal{A}$ is homeomorphic to $\mathbb{T}^1$ then $\overline{P}_{/\mathcal{A}}$ is orientation-preserving.*

Proof. Suppose that $\overline{P}_{/\mathcal{A}}$ is not orientation-preserving and consider the points A, B and the arcs C_+, C_- given by the last lemma. Consider any continuous parameterization of $\mathcal{A}$, $\alpha : [0,1] \to \mathcal{C}$ such that $\alpha(0) = \alpha(1) = A$, $\alpha(1/2) = B$, $\alpha(]0,1/2[) = C_+$, $\alpha(]1/2,1[) = C_-$, and the restriction of α to $]0,1[$ is one-to-one. Since $\mathcal{A}$ is not contractible in the cylinder, the curve α defines an element of the fundamental group of $\mathcal{C}$ different from the identity, say $[\alpha] \in \pi_1(C)$. The curve $\overline{P} \circ \alpha$ is homotopic to α via the homotopy $H_1 : [0,1] \times [0,T] \to \mathcal{C}$ defined by $H_1(t,\lambda) = \overline{x(\lambda; 0, \alpha(t))}$, so $\overline{P} \circ \alpha$ defines the same element in $\pi_1(C)$ as α. The curve

$$\gamma(t) = \begin{cases} \alpha(2t) \text{ if } t \in [0,1/2] \\ (\overline{P} \circ \alpha)(2t-1) \text{ if } t \in [1/2,1] \end{cases}$$

should defines an element different from the identity of $\pi_1(C)$, but it is homotopic to the point A via the homotopy $H_2 : [0,1] \times [0,1] \to \mathcal{C}$ defined by

$$H_2(t,\lambda) = \begin{cases} \gamma(t) \text{ if } t \in [0,\lambda/2] \\ \gamma(\lambda/2) \text{ if } t \in [\lambda/2, \psi(\lambda/2)] \\ \gamma(t) \text{ if } t \in [\psi(\lambda/2), 1] \end{cases},$$

where $\psi : [0,1/2] \to [1/2,1]$ is defined by

$$\psi(t) = \begin{cases} \frac{\alpha^{-1}(\overline{P}^{-1}(\alpha(2t)))+1}{2} \text{ if } t \in]0,1/2[\\ \psi(0) = 1, \ \psi(1/2) = 1/2 \end{cases}.$$

Which is a contradiction.

Given the last result we can define the rotation number of $\overline{P}_{/\mathcal{A}}$ as the element of $\mathbb{T}^1$

$$\rho(\overline{P}_{/\mathcal{A}}) = \lim_{n\to+\infty} \frac{f^n(\theta)}{n} + \mathbb{Z},$$

where $\theta \in \mathbb{R}$. The proof that $\rho(\overline{P}_{/\mathcal{A}})$ is well defined and does not depend on θ can be seen, for example, in [2] as well as the proof to the following proposition:

Proposition 5 *Suppose that $\mathcal{A}$ is homeomorphic to $\mathbb{T}^1$ and*

$$\rho(\overline{P}_{/\mathcal{A}}) = \frac{n}{m} + \mathbb{Z}, \tag{6}$$

where $n \in \mathbb{Z}$, $m \in \mathbb{N}$ and n/m is an irreducible fraction (if $n = 0$ then $m = 1$). Then, $\overline{P}_{/\mathcal{A}}$ has at least one periodic point of minimal period m. Moreover, any other periodic point of $\overline{P}_{/\mathcal{A}}$ has period m. Conversely, if $\overline{P}_{/\mathcal{A}}$ has a periodic point of period m then there exists $n \in \mathbb{Z}$ such that the rotation number of $\overline{P}_{/\mathcal{A}}$ is given by (6).

Consider an open, bounded, and convex set $\Omega \subset \mathbb{R}^2$ and the space $C^1(\overline{\Omega}, \mathbb{R}^2)$ with the norm

$$\|f\|_1 = \sum_{i=1}^{2} \sup_{\overline{\Omega}} \|f_i\| + \sum_{i=1,2,\ j=1,2} \sup_{\overline{\Omega}} \left\| \frac{\partial f_i}{\partial x_j} \right\|.$$

Lemma 6 *Consider the family*

$$P_\epsilon \in C^1(\overline{\Omega}, \mathbb{R}^2), \quad \epsilon \in [0,1],$$

such that if ϵ_n is a sequence in $[0,1]$ tending to ϵ_* then $P_{\epsilon_n} \to P_{\epsilon_*}$ in $C^1(\overline{\Omega}, \mathbb{R}^2)$. If $0 \in \Omega$, $P_0(0) = 0$ and $-1 \notin \sigma(P_0'(0))$, there exists $\epsilon_0 > 0$ and $Q > 0$ such that if $P_\epsilon^2(x) = x$ for some $\epsilon \in [0, \epsilon_0]$ and $\|x\| < Q$ then $P_\epsilon(x) = x$.

Proof. Suppose by contradiction that there exist sequences $\epsilon_n \to 0$ in $[0,1]$ and $x_n \to 0$ in $\mathbb{R}^2$ such that $P_{\epsilon_n}^2(x_n) = x_n$ but $P_{\epsilon_n}(x_n) \neq x_n$. We have

$$P_{\epsilon_n}(x_n) - P_{\epsilon_n}^2(x_n) = \left[\int_0^1 P_{\epsilon_n}'(tx_n + (1-t)P_{\epsilon_n}(x_n))dt\right](x_n - P_{\epsilon_n}(x_n)). \tag{7}$$

Now, define

$$z_n = \frac{P_{\epsilon_n}(x_n) - x_n}{\|P_{\epsilon_n}(x_n) - x_n\|}$$

and suppose that for a subsequence $z_n \to z$. Dividing (7) by $\|P_{\epsilon_n}(x_n) - x_n\|$ and passing to the limit yields $P_0'(0)z = -z$ which is a contradiction.

The computation of the index in the proof of the next Lemma uses some ideas from [11] and [3].

Lemma 7 *Suppose that y is an $(a,b)-$periodic, degenerate solution of (2) such that $y_0 = y(0)$ is an isolated fixed point of $P^b - aR$. Then $\gamma_b(y) \in \{-1,0,1\}$, y_0 is an isolated fixed point of $P^{2b} - 2aR$ and $\gamma_b(y) = \gamma_{2b}(y)$.*

Proof. Suppose that the characteristic multipliers of y are $\mu_1 = 1$ and $0 < \mu_2 < 1$. We shall assume that $y_0 = 0$, otherwise we do a translation. Let D be a linear and non-singular application from $\mathbb{R}^2$ to itself, such that

$$D(P^b - aR)'(0)D^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

Consider the function $F = D(P^b - aR)D^{-1}$. Writing the Taylor formula for F around the origin we obtain

$$F(x,y) = (x + R_1(x,y), \mu_2 y + R_2(x,y)),$$

where $R_i(x,y) = o(\|(x,y)\|)$, $i = 1,2$. Since $\frac{\partial(I-F)_2}{\partial y}(0,0) = (1-\mu_2) \neq 0$ we can apply the implicit function Theorem and conclude that there exists neighborhoods of the origin $V \subset \mathbb{R}$ and $W \subset \mathbb{R}^2$ and a C^1 function $\phi : V \to \mathbb{R}$ such that

$$\left\{ \begin{array}{l} (I-F)_2(x,y) = 0 \\ (x,y) \in W \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} y = \phi(x) \\ x \in V \end{array} \right. .$$

Consequently

$$\left\{ \begin{array}{l} (I-F)(x,y) = 0 \\ (x,y) \in W \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} R_1(x,\phi(x)) = 0 \\ y = \phi(x) \\ x \in V \end{array} \right. .$$

Let $\Phi : V \to \mathbb{R}$ be given by $\Phi(x) = R_1(x,\phi(x))$. Since 0 is an isolated fixed point of $P^b - aR$, 0 is an isolated root of Φ. We distinguish four types of behaviour of Φ around the origin. They are sketched in the following figure.
1^{o} Case: Define the family of functions

$$F_\epsilon(x,y) = ((1+\epsilon)x + R_1(x,y), \mu_2 y + R_2(x,y)),\ \epsilon \in [0,1].$$

Thus $F_0 = F$ and

$$\left\{ \begin{array}{l} (I-F_\epsilon)(x,y) = 0 \\ (x,y) \in W \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \Phi(x) = -\epsilon x \\ y = \phi(x) \\ x \in V \end{array} \right. \Leftrightarrow (x,y) = (0,0).$$

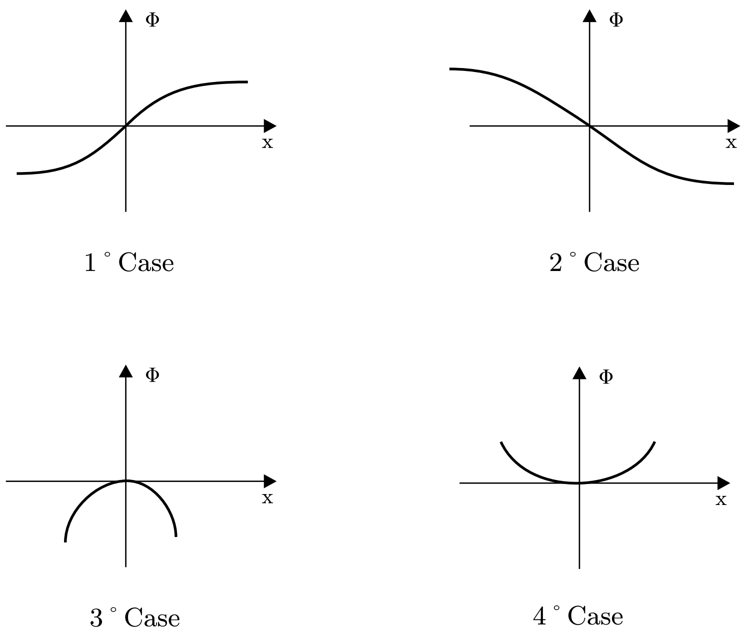

Figure 1:

i.e. F_ϵ is an admissible homotopy in any sufficiently small ball B_1 around the origin. We conclude that

$$\gamma_b(y) = deg(I - (P^b - aR), D^{-1}(B_1)) = deg(I - F, B_1)$$
$$= deg(I - F_1, B_1) = sign\{(1 - \mu_2)\} = 1.$$

Since $-1 \notin \sigma(F'(0))$, we can use the last Lemma to conclude that there exists $\epsilon_0 > 0$ and $Q > 0$ such that if $\epsilon \in [0, \epsilon_0]$ and $F_\epsilon^2(x) = x$ with $\|x\| < Q$ then $F_\epsilon(x) = x$. In particular 0 is an isolated fixed point of F^2. If $B_2 \subset B_1$ is a ball around the origin with radius smaller then Q we obtain

$$\gamma_{2b}(y) = deg(I - (P^b - aR)^2, D^{-1}(B_2)) = deg(I - F^2, B_2)$$
$$= deg(I - F_{\epsilon_0}^2, B_2) = sign\{(1 - \mu_2^2)(1 - (1 - \epsilon_0)^2)\} = 1.$$

2^{o} Case: Is similar to the 1^{o} Case.
3^{o} Case: We consider the family

$$F_\epsilon(x, y) = (-\epsilon x^2 + x + R_1(x, y), \mu_2 y + R_2(x, y)), \ \epsilon \in [0, 1].$$

Observing that $I - F_\epsilon$, $\epsilon > 0$, does not vanish in a small neighborhood of the origin, we have that F_ϵ is an admissible homotopy in a sufficiently small ball B_1 around the origin, so

$$\gamma_b(y) = deg(I - (P^b - aR), D^{-1}(B_1)) = deg(I - F, B_1) = deg(I - F_1, B_1) = 0.$$

On the other hand, by the last Lemma there exists $\epsilon_0 > 0$ and a small ball B_2 such that F_ϵ^2 does not vanish in $\mathcal{B}_2$ for any $\epsilon \in [0, \epsilon_0]$; so we conclude that

$$\gamma_{2b}(y) = deg(I - (P^b - aR)^2, D^{-1}(B_2)) = deg(I - F^2, B_2)$$

$$= deg(I - F_{\epsilon_0}^2, B_2) = 0.$$

4º Case: Similar to the 3º Case.

The differentiability of the Poincaré map utilized in the proof of the last Lemma seems to be essential due to the example constructed in [1].

The proof of the following Lemma can be seen in [8].

Lemma 8 *Consider the closed rectangle $\Sigma \subset \mathbb{R}^2$ with edges $d//e$ parallel to R and $f//f+R$ perpendicular to R (see fig 2). Suppose that $F = (F_1, F_2) : \Sigma \to \mathbb{R}^2$ is a continuous function such that*

$$F(x) = F(x+R) \neq 0, \ \forall x \in f;$$

$$< F(x), R >< 0, \ \forall x \in d;$$

$$< F(x), R >> 0, \ \forall x \in e.$$

Then $deg[F, \Sigma] = 0$.

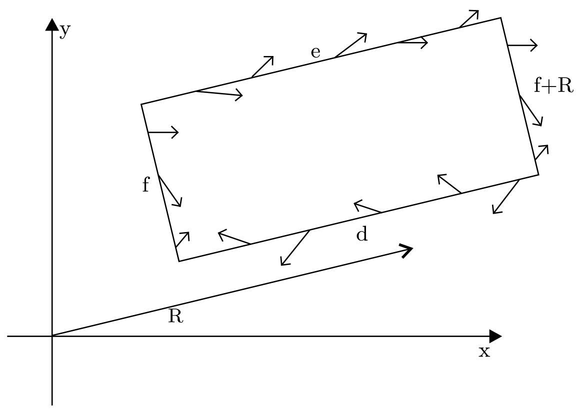

Figure 2:

Proposition 9 *Suppose that for some $(a, b) \in \mathbb{Z} \times \mathbb{N}$, $b \geq 1$, the set of $(a, b)-$periodic solutions of (2) is finite and given by*

$$y_1, \ y_2, \ldots, y_n.$$

Then

$$\sum_{i=1}^{n} \gamma_b(y_i) = 0.$$

Proof. Consider the rectangle

$$\Sigma = \{x \in \mathbb{R}^2 : \alpha \leq < x, R > \leq \alpha + 1 \text{ and } | < x, (-\overline{h}, 1) > | \leq \rho\},$$

where, α and ρ are positive constants. Suppose that

$$y_i(0) \in int(\Sigma), \ i = 1, 2, \ldots, n.$$

It is not difficult to prove that if ρ is sufficiently large, Σ and the function $F = I - (P^b - aR)$ satisfy the hypothesis of the last Lemma; so we conclude that $deg[I - (P^b - aR), \Sigma] = 0$. Now, if $\Omega_i, \ i = 1, \ldots, n$ are disjoint open sets of Σ such that

$$\bigcup_i \overline{\Omega}_i = \Sigma \text{ and } y_i(0) \in \Omega_i, \ i = 1, \ldots, n;$$

we have

$$\sum_{i=1}^{n} \gamma_b(y_i) = \sum_{i=1}^{n} deg[I - (P^b - aR), \Omega_i] = deg[I - (P^b - aR), \Sigma] = 0,$$

as pretended.

Lemma 10 *Suppose that for some $(a, b) \in \mathbb{Z} \times \mathbb{N}$, $b \geq 1$, the set of $(a, b)-$periodic solutions of (2) is finite. If there exists an inversely unstable $(a, b)-$periodic solution then there exists an $(2a, 2b)-$periodic solution which is not $(a, b)-$periodic.*

Proof. Suppose by contradiction that all the $(2a, 2b)-$periodic solutions are $(a, b)-$periodic. Applying the last Lemma twice yields

$$\sum_{y \text{ is } (a,b)-\text{periodic}} \gamma_b(y) = 0 = \sum_{y \text{ is } (2a,2b)-\text{periodic}} \gamma_{2b}(y). \tag{8}$$

Moreover, each of the above sums has the same number of elements. Notice that, if the characteristic multipliers μ_1 and μ_2 are real numbers:

- If $0 < \mu_1 < 1 < \mu_2$ we have $\gamma_b(y) = -1$ and $\gamma_{2b}(y) = -1$.
- If $0 < \mu_1 < 1 = \mu_2$ by the Lemma 7 we have $\gamma_b(y) = \gamma_{2b}(y)$.

- If $0 < \mu_1 \leq \mu_2 < 1$ we have $\gamma_b(y) = 1$ and $\gamma_{2b}(y) = 1$.
- If $-1 < \mu_1 \leq \mu_2 < 0$ we have $\gamma_b(y) = 1$ and $\gamma_{2b}(y) = 1$.
- If $-1 = \mu_1 < \mu_2 < 0$ we have $\gamma_b(y) = 1$ and, by Lemma 7, $\gamma_{2b}(y) \in \{1, 0, -1\}$.
- If $\mu_1 < -1 < \mu_2 < 0$ we have $\gamma_b(y) = 1$ and $\gamma_{2b}(y) = -1$.

On the other hand, if μ_1 and μ_2 are complex then they are conjugate and we have $\gamma_b(y) = 1$ and $\gamma_{2b}(y) = 1$. We conclude that $\gamma_b(y) \geq \gamma_{2b}(y)$ for all solutions y of (2). Moreover, Lemma 7 shows that $\gamma_b(y), \gamma_{2b}(y) \in \{-1, 0, 1\}$ for all (a, b)-periodic solutions of (2). By (8) we conclude that $\gamma_b(y_i) = \gamma_{2b}(y_i),\ i = 1, \ldots, n$ for each (a, b)-periodic solution of (2); which is a contradiction with the existence of an inversely unstable solution.

Finally we can prove Theorem 1.

Proof of Theorem 1. If y is the inversely unstable (a, b)-periodic solution of (2) then $\overline{y(0)}$ is a periodic point of $\overline{P}$. On the other hand, by the last Lemma there exists a $(2a, 2b)-$periodic solution y_1 of (2) which is not (a, b)-periodic. Thus $\overline{y_1(0)}$ is a periodic point of $\overline{P}$ of period $2b$ but it is not of period b. Necessarily $(\overline{y(0)}, \overline{y_1(0)}) \in \mathcal{A}$ are periodic points of different minimal period. The Lemma 5 shows that the rotation number is not well defined and consequently $\mathcal{A}$ is not homeomorphic to $\mathbb{T}^1$.

3 The existence of inversely unstable solutions

The main goal of this section is the proof of Theorem 2. We shall start by using the ideas from [9] to construct a convenient linear equation that will be the linearized equation of the final example. Given positive constants w_1, w_2, and $k \in \mathbb{N}$, we define the kT-periodic step function by

$$\alpha(t) = \begin{cases} -w_1^2 + \frac{c^2}{4} \text{ if } t \in [0, \frac{kT}{2}] \\ w_2^2 + \frac{c^2}{4} \text{ if } t \in]\frac{kT}{2}, kT] \end{cases},$$

and consider the linear equation

$$\begin{cases} x_1' = x_2 - cx_1 \\ x_2' = -\alpha(t)x_1 \end{cases}. \tag{9}$$

If μ_1 and μ_2 are the characteristic multipliers of this equation, we define the discriminant as

$$\Delta_{(9)}[\alpha] = \mu_1 + \mu_2 = tr\Phi(kT),$$

where Φ is the fundamental matrix of (9) such that $\Phi(0) = I_2$. It is easy to see that (9) is inversely unstable iff

$$\Delta_{(9)}[\alpha] < -(1 + e^{-ckT}). \tag{10}$$

Observe that the change of variables $y_1 = e^{\frac{c}{2}t}x_1$, $y_2 = y_1'$, transform equation (9) in

$$\begin{cases} x_1' = x_2 \\ x_2' = -\left(\alpha(t) - \frac{c^2}{4}\right) x_1 \end{cases}. \tag{11}$$

If $\Delta_{(11)}[\alpha]$ is the discriminant of (11) then $\Delta_{(11)}[\alpha] = e^{\frac{c}{2}kT}\Delta_{(9)}[\alpha]$, so (9) is inversely unstable iff

$$\Delta_{(11)}[\alpha] < -2\cosh\left(\frac{c}{2}kT\right). \tag{12}$$

By direct computations we obtain

$$\Delta_{(11)}[\alpha] \tag{13}$$

$$= 2\cosh\left(\frac{w_1 kT}{2}\right)\cos\left(\frac{w_2 kT}{2}\right) + \sinh\left(\frac{w_1 kT}{2}\right)\sin\left(\frac{w_2 kT}{2}\right)\left(\frac{w_1}{w_2} - \frac{w_2}{w_1}\right).$$

Lemma 11 *Given $\mathcal{H} > c^2/4$ there exists w_1, w_2 and k such that equation (9) is inversely unstable and $\alpha < \mathcal{H}$.*

Proof. Let w_2 be such that $w_2 kT = 2\pi$. By (13), $\Delta_{(11)}[\alpha] = -2\cosh(w_1 kT/2)$. So by (12), equation (9) is inversely unstable iff $w_1 > c$. We fix an $w_1 > c$. Finally we choose a sufficiently large k such that $w_2^2 < \mathcal{H} - c^2/2$ in order that $\alpha < \mathcal{H}$.

Lemma 12 *Given $\mathcal{H} > c^2/4$ there exists $g \in C^\infty(\mathbb{R}/\mathbb{Z})$ and $p \in C^\infty(\mathbb{R}/kT\mathbb{Z})$ such that $g' < \mathcal{H}$ and equation (5) has an inversely unstable $(0,k)-$periodic solution.*

Proof. Given $\mathcal{H} > c^2/4$ consider the step function α given by the last Lemma. Applying Lemma 2.1 in [10] we can perform an approximation argument and assume that $\alpha \in C^\infty$. So we can fix a function $\alpha \in C^\infty(\mathbb{R}/kT\mathbb{Z})$ so that equation (9) is inversely unstable and $\alpha < \mathcal{H}$.

Consider constants $A, B \in \mathbb{R}$ so that $A < \alpha(t) < B < \mathcal{H}$. Let us fix a function $g \in C^\infty(\mathbb{R}/\mathbb{Z})$ in such a way that there exists an interval $I \subset [0,1]$ so that $g'' > 0$ in I and $g'(I) = [A, B]$. Finally, define $kT-$periodic functions $x(t) = (g'_{/I})^{-1}(\alpha(t))$ and $p(t) = x''(t) + cx'(t) + g(x(t))$. Thus $(x, x' + cx)$ is a solution of (5) and the linearized equation is (9) which by construction is inversely unstable; i.e. x is inversely unstable in the sense of Levinson and consequently it is inversely unstable under our definition.

In order to apply Theorem 1 is necessary to find an equation with a finite number of $(0,k)$-periodic solutions. Notice that the existence of an infinite number of $(0,k)$-periodic solutions implies the existence of an infinite number of fixed points of P^k with an accumulation point y_0. The point y_0 is fixed by P^k, so the solution with initial condition y_0 is $(0,k)$-periodic and degenerate. Hence it will be sufficient to exclude the existence of degenerate $(0,k)$-periodic solutions. We will show that, as a consequence of the Sard-Smale's Theorem, the set of forcings p for which equation (5) does not have degenerate $(0,k)$-periodic solutions is dense in $C(\mathbb{R}/kT\mathbb{Z})$.

For each $j \in \mathbb{N}$ consider the Banach space $C^j(\mathbb{R}/kT\mathbb{Z})$ with the norm

$$\|u\|_j = \sum_{i=0}^{j} \sup_{t\in[0,kT]} |u^{(i)}(t)|.$$

Let

$$A = \{p \in C(\mathbb{R}/kT\mathbb{Z}) \ / \ (5) \text{ has a degenerate } (0,k)\text{-periodic solution } \}.$$

Lemma 13 *The set $C(\mathbb{R}/kT\mathbb{Z}) \setminus A$ is dense in $C(\mathbb{R}/kT\mathbb{Z})$.*

Proof. The operator

$$\Phi : C^2(\mathbb{R}/kT\mathbb{Z}) \to C(\mathbb{R}/kT\mathbb{Z}), \quad \Phi(x) = x'' + cx' + g(x)$$

is of class C^1, Fredholm with index zero, and $\Phi'_u(h) = h'' + ch' + g'(u(t))h$. If $p \in C(\mathbb{R}/kT\mathbb{Z})$ is such that equation (5) has a degenerate $(0,k)$-periodic solution $u = (u_1, u_2)$ then 1 is a characteristic multiplier of

$$\begin{cases} \xi' = \eta - c\xi \\ \eta' = -g'(u_1(t))\xi \end{cases};$$

by Floquet theory there exists a nontrivial kT-periodic solution (ξ, η) of the last equation. Thus $\Phi'_{u_1}(\xi) = \Phi'_{u_1}(0)$, i.e. Φ'_{u_1} is not one-to-one and consequently $\Phi(u_1) = p$ is not a regular value of Φ. We conclude that

$$\{p \in C(\mathbb{R}/kT\mathbb{Z}) \ / \ p \text{ is a regular value of } \Phi\} = C(\mathbb{R}/kT\mathbb{Z}) \setminus A.$$

The result is now a consequence of Sard-Smale's Theorem.

We can finally prove Theorem 2.

Proof of Theorem 2. Consider the functions $g \in C^\infty(\mathbb{R}/\mathbb{Z})$, $p \in C^\infty(\mathbb{R}/kT\mathbb{Z})$ and the $(0,k)$-periodic solution $x = (x_1, x_2)$ given by Lemma 12. We have $\Phi(x_1(t)) = p$. Since the linearized equation

$$\begin{cases} \xi' = \eta - c\xi \\ \eta' = -g'(x_1(t))\xi \end{cases}$$

is inversely unstable, it does not have non-trivial kT-periodic solutions. Fredholm alternative shows that $h'' + ch' + g'(x_1(t))h = b(t),\ b \in C(\mathbb{R}/kT\mathbb{Z})$ has an unique $kT-$periodic solution, i.e. Φ'_{x_1} is an isomorphism. By the inverse function Theorem, Φ is a difeomorphism in a neighborhood of x_1. Using the last Lemma we can take a sequence p_n tending to p in $C(\mathbb{R}/kT\mathbb{Z})$ such that the equation

$$\begin{cases} y_1' = y_2 - cy_1 \\ y_2' = -g(y_1(t)) + p_n(t) \end{cases}$$

does not have degenerate kT-periodic solutions. For sufficiently large n we can define the sequence

$$(x_1)_n = \Phi^{-1}(p_n).$$

For each $n \in \mathbb{N}$, the function $z_n = ((x_1)_n, (x_1)'_n + c(x_1)_n)$ is a $(0,k)$-periodic solution of

$$\begin{cases} x_1' = x_2 - cx_1 \\ x_2' = -g(x_1) + p_n(t) \end{cases} \tag{14}$$

with linearized equation

$$\begin{cases} \xi' = \eta - c\xi \\ \eta' = -g'((x_1)_n(t))\xi \end{cases} . \tag{15}$$

Since, by Lebesgue's bounded convergence theorem, $g'((x_1)_n) \to g'(x_1)$ in the weak* topology, Lemma 2.1 in [10] shows that for $n = n_0$ sufficiently large (15) is inversely unstable. Since equation (14) does not have degenerate $(0,k)-$periodic solutions , z_{n_0} is isolated and consequently it is an inversely unstable solution. The result is now consequence of Theorem 1.

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