Bounds for non periodic mixtures of infinitely many materials

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Summary We prove bounds on the homogenized coefficients for general non periodic mixtures of an arbitrary number of isotropic materials, in the heat conduction framework. The component materials and their proportions are given through the Young measure associated to the sequence of coefficient functions. Upper and lower bounds inequalities are deduced in terms of algebraic relations between this Young measure and the eigenvalues of the H-limit matrix. The proofs employ arguments of compensated compactness and fine properties of Young measures. When restricted to the periodic case, we recover known bounds.

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1. State of the art

Homogenization theory has developed following two approaches: the general case (under no additional hypothesis on the material coefficients) and the periodic case (assuming periodicity of the coefficients). The periodicity assumption is restrictive from the theoretical point of view but most numerical experiences are confined to the periodic framework. Also the periodicity assumption allowed the extention of the homogenization theory to more general frameworks like non linear elasticity, p-laplacian, homogenization of general energy functionals.

A problem of major importance in homogenization is the identification of bounds on the homogenized coefficients, known in the literature as the G-closure problem. It consists in describing the set of all possible tensors obtained by mixing a given number of base materials, in given proportions. Once identified such optimal bounds on homogenized coefficients, one is able to choose the composite material which fits best to a given goal. This is called optimization via homogenization, a field which developes as an important direction of shape optimization. That is why the problem of optimal bounds is crucial for continuum mechanics of composite media. The first results on this subject are due to A. Cherkaev, K. Lurie (see [9], [10]), F. Murat, L. Tartar (see [26], [31]) and do not assume any periodicity assumptions. G. Francfort, F. Murat in [11] generalized these bounds for isotropic mixtures of elastic materials.

Afterwards, several generalizations of these results have been obtained for periodic mixtures of materials. G. Allaire, R. Kohn, Y. Grabovsky, L. Gibiansky (see [1 – 4], [12], [13 – 15]) proved optimal bounds on the elastic energy of two phase periodic composites. In the heat conduction framework, G. Milton, R. Kohn (see [24]), V. Nesi (see [27 – 29]) obtained bounds on the homogenized coefficients for periodic mixtures of several isotropic materials. Bounds for different kinds of problems have also been deduced under periodicity assumptions : for thin structures G. Bouchitté, G. Buttazzo, I. Fragalà [8], for composites with interface between materials R. Lipton [17 – 19], for the p-Laplacian D. Lukkassen, L.E. Persson, P. Wall [20], for non linear materials P. Wall [34].

In the present paper we extend the results obtained under periodicity assumptions by C. Barbarosie [7] to the general non periodic case. We consider mixtures made of infinitely many isotropic materials, in the heat conduction framework. The component materials and their proportions are given through the Young measure associated to the sequence of coefficient functions. The bounds are described by the relationship between this Young measure and the respective H-limit matrix.

In Section 2 we give a brief survey on homogenization theory and a general description of the problem under consideration. Our main results are Theorems 3.2, 3.3 and 3.5 stated and proven in Section 3. For two dimensional lower bounds the proofs are performed with all details while for the upper bounds and for the n-dimensional lower bounds we just point out the essential steps of the proofs. Section 4 is devoted to some key lemmas on which the main results are based. Useful results on Young measures, regularization, linear algebra and quadratic forms are contained in Appendices A, B and C.

2. Setting of the problem

Let us briefly present some notions of homogenization theory following the guidelines in [25] and [26].

Let Ω be a bounded domain in $\mathbb{R}^n$, with $n \geq 2$. Consider a sequence of conductivity tensors (A_ε) and a sequence of temperature functions (u_ε) satisfying the equation $-div A_\varepsilon \nabla u_\varepsilon = f$ in Ω. The index ε represents a small parameter converging to zero. Note that the heat source f does not oscillate (is independent of ε). Assume that the sequence of temperature gradients (∇u_ε) converges weakly in $L^2(\Omega; \mathbb{R}^n)$ to a vector field $\vec{g}_0$. Assume also that the sequence of heat fluxes $(\vec{\jmath}_\varepsilon = -A_\varepsilon \nabla u_\varepsilon)$ converges weakly to the vector field $\vec{\jmath}_0$. Then the matrix A_H of effective thermal coefficients (which describe the macroscopic behaviour of the mixture) is implicitly defined by the equality $\vec{\jmath}_0 = -A_H \vec{g}_0$.

Consider two real positive numbers α e β, with $\alpha < \beta$. Define the set $\mathcal{M}(\alpha, \beta)$ of matrices A, $n \times n$ symmetric and such that $\alpha I \leq A \leq \beta I$. Define also the set $\mathcal{M}(\alpha, \beta, \Omega)$ of measurable functions $A : \Omega \to \mathcal{M}(\alpha, \beta)$.

Theorem 2.1. *Let (A_ε) be a sequence of functions in $\mathcal{M}(\alpha, \beta, \Omega)$, and $A_H \in \mathcal{M}(\alpha, \beta, \Omega)$. Then the following two statements are equivalent:*
a) For all $f \in H^{-1}(\Omega)$ and for all $\bar{u} \in H^{1/2}(\partial\Omega)$, the solution $u_\varepsilon \in H^1(\Omega)$ (which exists and is unique) of the problem

$$\begin{cases} -div(A_\varepsilon \nabla u_\varepsilon) = f & in\ \Omega \\ u_\varepsilon = \bar{u} & on\ \partial\Omega \end{cases}$$

converges, weakly in $H^1(\Omega)$, to the solution $u_0 \in H^1(\Omega)$ (which exists and is unique) of the problem

$$\begin{cases} -div(A_H \nabla u_0) = f & in\ \Omega \\ u_0 = \bar{u} & on\ \partial\Omega \end{cases}$$

b) For any sequence of vector filelds $(\vec{g}_\varepsilon)$ that converges (weakly in $L^2(\Omega; \mathbb{R}^n)$) to $\vec{g}_0$, such that the sequence of vector fields $(\vec{\jmath}_\varepsilon)$, with $\vec{\jmath}_\varepsilon = -A_\varepsilon \vec{g}_\varepsilon$, converges (weakly in $L^2(\Omega; \mathbb{R}^n)$) to $\vec{\jmath}_0$, and such that the sequences $(rot\vec{g}_\varepsilon)$ and $(div\vec{\jmath}_\varepsilon)$ are relatively compact in $H^{-1}(\Omega)$, the following equality holds $\vec{\jmath}_0 = -A_H \vec{g}_0$.

Note that, in order for a sequence of functions to be relatively compact in $H^{-1}(\Omega)$, it is sufficient to be bounded in $L^2(\Omega)$. Note also that the expression $rot\vec{g}$ represents the set of differences of cross derivatives $\dfrac{\partial g_i}{\partial x_j} - \dfrac{\partial g_j}{\partial x_i}$, for $1 \le i < j \le n$.
The notion of convergence in the set of conductivity tensors, apropriated to the study of composite materials, is given as follows:

Definition 2.2. *Under the equivalent hypothesis a) or b) stated in Theorem 2.1, one says that A_H is the H-limit of the sequence (A_ε) (or, the limit in the sense of the homogenization theory), and writes $A_\varepsilon \xrightarrow{H} A_H$.*

The H-convergence above described derives from a topology on the space $\mathcal{M}(\alpha, \beta, \Omega)$. Moreover, this topology is metrizable, i.e. there exists a distance on $\mathcal{M}(\alpha, \beta, \Omega)$ defining the H-convergence topology.
The following three results state fundamental properties of the H-convergence which are important for the problem under consideration.

Theorem 2.3.(compacity) *Consider a sequence of functions (A_ε) in $\mathcal{M}(\alpha, \beta, \Omega)$. Then there exists a subsequence ε' of ε and a matrix function $A_H \in \mathcal{M}(\alpha, \beta, \Omega)$ such that $A_{\varepsilon'} \xrightarrow{H} A_H$.*

Theorem 2.4.(convergence of the energy) *Consider a sequence of functions (A_ε) in $\mathcal{M}(\alpha, \beta, \Omega)$, which H-converges to A_H. Consider a sequence of vector fields $(\vec{g}_\varepsilon)$ satisfying the condition b) in the satement of Theorem 2.1. Then the sequence $(\langle A_\varepsilon \vec{g}_\varepsilon, \vec{g}_\varepsilon \rangle)$,* *where $\langle A_\varepsilon \vec{g}_\varepsilon, \vec{g}_\varepsilon \rangle = -\langle \vec{\jmath}_\varepsilon, \vec{g}_\varepsilon \rangle$, converges in the sense of distributions to $\langle A_H \vec{g}_0, \vec{g}_0 \rangle$, where $\langle A_H \vec{g}_0, \vec{g}_0 \rangle = -\langle \vec{\jmath}_0, \vec{g}_0 \rangle$.*

Theorem 2.5.(local character of H-convergence) *Let Ω_1 and Ω_2 be two domains in $\mathbb{R}^n$, and let (A_ε) and (B_ε) be two sequences of matrix functions on Ω_1, respectively on Ω_2 such that $A_\varepsilon \xrightharpoonup{H} A_H$ and $B_\varepsilon \xrightharpoonup{H} B_H$. If $A_\varepsilon(x) = B_\varepsilon(x)$ for almost every $x \in \Omega_1 \cap \Omega_2$ and for all $\varepsilon > 0$, then $A_H(x) = B_H(x)$ for almost every $x \in \Omega_1 \cap \Omega_2$.*

Our goal is to describe the set of all matrices A_H which can be obtained as H-limit of a sequence of isotropic materials when the component materials and their proportions are given. Namely, let (a_ε) be a sequence in $L^\infty(\Omega)$ and consider the sequence of isotropic conductivity matrices $(a_\varepsilon I)$ (here I is the $n \times n$ identity matrix). By the compacity result (Theorem 2.3) there exists a subsequence of ε (which we shall still denote by ε) such that $(a_\varepsilon I)$ H-converges to some matrix function A_H. On the other hand, by extracting another subsequence of ε, still denoted by ε, there exists a Young measure μ associated to (a_ε) (see Remarks A.1 2) and A.2 2)).

The Young measure μ contains all the information about the component materials and the proportions in which they are used. For each point $x \in \Omega$, the slice μ_x of the disintegration of $\mu = (\mu_x)_{x\in\Omega}$ (see Appendix A) describes the composition of the mixture at the point x. The support of the probability μ_x (which is a subset of the interval $[\alpha, \beta]$) represents the set of conductivity coefficients of the component materials. For a borelian set $B \subset [\alpha, \beta]$, $\mu_x(B)$ represents the proportion in which materials whose coefficients belong to B are used.

For instance, let us consider the known case of binary mixtures. Consider two base materials of conductivity coefficients α and β. Suppose that we have at our disposal a proportion θ of material α and a proportion $1-\theta$ of material β ($0 \le \theta \le 1$). We build finer and finer mixtures of these two materials in the given proportions. Mathematically this process writes as $a_\varepsilon(x) = \chi_\varepsilon(x)\alpha + (1-\chi_\varepsilon(x))\beta$ where (χ_ε) is a sequence of characteristic functions such that $\chi_\varepsilon \rightharpoonup \theta$ in the weak $*$ topology of $L^\infty(\Omega)$. Actually we consider θ as a function in $L^\infty(\Omega; [0,1])$ and consequently in each point x of the domain Ω we have a mixture of α and β in the proportions $\theta(x)$ and $1-\theta(x)$, respectively. In this case each slice of the disintegration of the Young measure μ is a convex combination of Dirac masses : $\mu_x = \theta(x)\delta_\alpha + (1-\theta(x))\delta_\beta$.

We characterize the set of attainable matrices A_H in terms of the Young measure μ.

We use several different ingredients like compensated compactness introduced by L. Tartar in [32], classical minimization of quadratic functionals on affine spaces and a technique inspired in the scale convergence tool introduced by L. Mascarenhas and A.-M. Toader in [22]. This latest technique consists in relating Young measures associated to sequences of vector functions to the Young measures of their components.

Upper and lower bounds are deduced for the matrices A_H. We obtain algebraic relations which involve the eigenvalues of A_H and which can be written in terms of a determinant.

Namely, for the lower bound in two dimensions ($n = 2$) :

$$\det \begin{bmatrix} \lambda_1(x) - a_*(x) & \alpha - a_*(x) \\ \alpha - a_*(x) & \lambda_2(x) - a_*(x) \end{bmatrix} \geq 0$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are the eigenvalues of $A_H(x)$, while $a_*(x)$ is defined by

$$\frac{1}{a_*(x)} = \int_{[\alpha,\beta]} \frac{2}{a+\alpha} d\mu_x(a)$$

(see Theorem 3.2). For the n dimensional case ($n \geq 2$) the lower bound is obtained in Theorem 3.5 and writes like :

$$\det \begin{bmatrix} \lambda_1(x) - a_*^{(m)}(x) & \alpha - a_*^{(m)}(x) & \dots & \alpha - a_*^{(m)}(x) \\ \alpha - a_*^{(m)}(x) & \lambda_2(x) - a_*^{(m)}(x) & \dots & \alpha - a_*^{(m)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha - a_*^{(m)}(x) & \alpha - a_*^{(m)}(x) & \dots & \lambda_m(x) - a_*^{(m)}(x) \end{bmatrix} \geq 0$$

where $\lambda_1(x), \lambda_2(x), \dots, \lambda_m(x)$ are any m eigenvalues of $A_H(x)$, with $1 \leq m \leq n$, while $a_*^{(m)}(x)$ is defined by

$$\frac{1}{a_*^{(m)}(x)} = \int_{[\alpha,\beta]} \frac{m}{a+(m-1)\alpha} d\mu_x(a).$$

Upper bounds are deduced, only for the two dimensional case, in the form:

$$\det \begin{bmatrix} \lambda_1^{-1}(x) - a_{**}^{-1}(x) & \beta^{-1} - a_{**}^{-1}(x) \\ \beta^{-1} - a_{**}^{-1}(x) & \lambda_2^{-1}(x) - a_{**}^{-1}(x) \end{bmatrix} \geq 0$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are the eigenvalues of $A_H(x)$, while $a_{**}(x)$ is defined by

$$a_{**}(x) = \int_{[\alpha,\beta]} \frac{2}{a^{-1}+\beta^{-1}} d\mu_x(a)$$

(see Theorem 3.3).

The optimality of these bounds can be proven under the same supplementary hypotheses like in the periodic framework (see Remark 3.4).

3. Main results

Consider R a rotation operator in $\mathbb{R}^2$ (i. e. a 2×2 matrix with the properties $R^2 = -I$, $R^T = -R$). Typically, $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Theorem 3.2 below states the lower bound inequality in the two dimensional case ($n = 2$). In order to be more explicit we splitted the proof in two parts : an intermediate result (Proposition 3.1) and the theorem itself (Theorem 3.2). The proof of the following proposition uses techniques of compensated compactness from [32] and Young measures. The idea of introducing the Young measure associated to a triplet and to relate it to the Young measures of the components appeared in [22] where it was used for setting the notion of scale convergence.

Proposition 3.1. *Consider $\Omega \subset \mathbb{R}^2$ a bounded domain, with boundary of class $\mathcal{C}^q$ for some $q > 2$, and let (a_ε) be a sequence of functions in $L^\infty(\Omega, [\alpha, \beta])$. Consider A_H the H-limit of $(a_\varepsilon I)$ and μ the Young measure associated to (a_ε). Suppose that $\mu_x(\{\alpha\}) > 0$ for almost every $x \in \Omega$ (in the sense of the Lebesgue measure). For almost every $x \in \Omega$, define $a_*(x)$ by*

$$\frac{1}{a_*(x)} = \int_{[\alpha,\beta]} \frac{2}{a+\alpha} d\mu_x(a).$$

Let u_0 and v_0 be two functions in $\mathcal{C}^1(\Omega)$ with compact support. Then, for almost every $x \in \Omega$, the following inequality holds

$$\begin{aligned} &\langle A_H(x)\nabla u_0(x), \nabla u_0(x)\rangle + \langle A_H(x)\nabla v_0(x), \nabla v_0(x)\rangle + \\ &\quad + 2\alpha\langle \nabla u_0(x), R\nabla v_0(x)\rangle \geq a_*(x) \left|\nabla u_0(x) + R\nabla v_0(x)\right|^2. \end{aligned}$$

Proof: In order to clarify the need of making regulatity assumptions on the boundary of Ω and on the two functions u_0 and v_0, let us drop out these assumptions for a while. We take Ω to be an arbitrary bounded open subset of $\mathbb{R}^2$ and take two fixed functions $u_0, v_0 \in H_0^1(\Omega)$. It is obvious that u_0 is the unique solution of the problem

$$\begin{gathered} \text{find } u \in H_0^1(\Omega) \text{ such that} \\ -div(A_H \nabla u) = -div(A_H \nabla u_0). \end{gathered}$$

Thus, by taking the fixed source term $-div(A_H\nabla u_0) \in H^{-1}(\Omega)$ in Theorem 2.1, we conclude that the sequence of functions (u_ε) defined by

$$\begin{gathered} u_\varepsilon \in H_0^1(\Omega) \\ -div(a_\varepsilon \nabla u_\varepsilon) = -div(A_H \nabla u_0) \end{gathered} \tag{3.1}$$

converges weakly in $H^1(\Omega)$ to u_0. Similarly, by taking the fixed source term $-div(A_H\nabla v_0) \in H^{-1}(\Omega)$, we conlcude that the sequence of functions (v_ε) defined by

$$\begin{gathered} v_\varepsilon \in H_0^1(\Omega) \\ -div(a_\varepsilon \nabla v_\varepsilon) = -div(A_H \nabla v_0) \end{gathered} \tag{3.2}$$

converges weakly in $H^1(\Omega)$ to v_0.

The sequences (∇u_ε) and (∇v_ε) are weakly convergent (and consequently bounded) in $L^2(\Omega)$, thus the products $(\langle a_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon\rangle)$, $(\langle a_\varepsilon \nabla v_\varepsilon, \nabla v_\varepsilon\rangle)$ and $(\langle \nabla u_\varepsilon, R\nabla v_\varepsilon\rangle)$ are bounded sequences in $L^1(\Omega)$.

The following weak convergences hold in the sense of distributions:

$$\begin{aligned} \langle a_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon \rangle &\rightharpoonup \langle A_H \nabla u_0, \nabla u_0 \rangle, \\ \langle a_\varepsilon \nabla v_\varepsilon, \nabla v_\varepsilon \rangle &\rightharpoonup \langle A_H \nabla v_0, \nabla v_0 \rangle, \\ \langle \nabla u_\varepsilon, R \nabla u_\varepsilon \rangle &\rightharpoonup \langle \nabla u_0, R \nabla v_0 \rangle. \end{aligned} \tag{3.3}$$

The first two limits are known in homogenization theory as "convergence of energy"(see Theorem 2.4). Having in mind that $rot \nabla u_\varepsilon = 0$ and that $div R \nabla v_\varepsilon = 0$, the third convergence turns out to be a consequence of the div-curl lemma (see [32]).

We need to relate the weak limits appearing above with the Young measure μ. This link passes through the Young measure θ associated to the triplet $(a_\varepsilon, \nabla u_\varepsilon, \nabla v_\varepsilon)$. The idea of introducing such a Young measure appeared in [22] where it was used for setting the notion of scale convergence. So, θ is a positive measure on $\Omega \times [\alpha, \beta] \times \mathbb{R}^2 \times \mathbb{R}^2$ whose projection on Ω is the Lebesgue measure. We also know that the projection of θ on $\Omega \times [\alpha, \beta]$ is equal to μ. In particular, $\mu_x(\{\alpha\}) > 0$ implies $\theta_x(\{\alpha\} \times \mathbb{R}^2 \times \mathbb{R}^2) > 0$ for almost every $x \in \Omega$.

From the definition of a Young measure associated to a sequence of functions (see Remark A.2 2), one can pass to the limit expressions like $\varphi(a_\varepsilon, \nabla u_\varepsilon, \nabla v_\varepsilon)$, where $\varphi : [\alpha, \beta] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is any continuous bounded function (we do not need the x variable here). But we cannot use this property for expressions like $\langle a_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon \rangle$, because the function $(a, \xi, \eta) \mapsto \langle a\xi, \xi \rangle$ is not bounded on $[\alpha, \beta] \times \mathbb{R}^2 \times \mathbb{R}^2$. This difficulty can be overpassed by checking the uniform integrability of the sequence of functions $\langle a_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon \rangle$ and by applying Theorem A.3. However, the sequence $(\langle a_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon \rangle)$ is not uniformly integrable in general (it is just a bounded sequence in $L^1(\Omega)$).

The regularity assumptions on the boundary of Ω ($\mathcal{C}^q$ for some $q > 2$) and on the functions u_0 and v_0 ($\mathcal{C}^1$ with compact support) allow us to overcome the difficulties described above by using a result of N. Meyers [23]. Theorem B (see the Appendix) ensures the existence of an exponent $p_0 > 2$ (depending only on the domain Ω and on the coercivity ratio β/α) such that, for any $p \in]2, p_0[$, if $A_H \nabla u_0$ belongs to $L^p(\Omega)$ then ∇u_ε belongs to a bounded subset of $L^p(\Omega)$ (see equation (3.1) which defines u_ε). If we take $u_0 \in \mathcal{C}^1(\Omega)$ with compact support, then $A_H \nabla u_0$ belongs to $L^\infty(\Omega)$ and thus to $L^p(\Omega)$ for any $p > 2$, the same happening for $A_H \nabla v_0$. We conclude that the sequences (∇u_ε) and (∇v_ε) are bounded in $L^p(\Omega)$ for any $p \in]2, p_0[$, hence the three products appearing in (3.3) are bounded sequences in $L^{p/2}(\Omega)$, for some $p > 2$. As any function in $L^q(\Omega)$, with $q > 1$, is uniformly integrable, it turns out that the sequences in (3.3) are uniformly integrable. Thus, we can apply Theorem A.3 and pass to the weak limit in $L^2(\Omega)$ as follows:

$$\begin{aligned} \langle a_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon \rangle &\rightharpoonup \int_{[\alpha,\beta] \times \mathbb{R}^2 \times \mathbb{R}^2} \langle a\xi, \xi \rangle \, d\theta_x(a, \xi, \eta), \\ \langle a_\varepsilon \nabla v_\varepsilon, \nabla v_\varepsilon \rangle &\rightharpoonup \int_{[\alpha,\beta] \times \mathbb{R}^2 \times \mathbb{R}^2} \langle a\eta, \eta \rangle \, d\theta_x(a, \xi, \eta), \\ \langle \nabla u_\varepsilon, R \nabla v_\varepsilon \rangle &\rightharpoonup \int_{[\alpha,\beta] \times \mathbb{R}^2 \times \mathbb{R}^2} \langle \xi, R\eta \rangle \, d\theta_x(a, \xi, \eta). \end{aligned} \tag{3.4}$$

Consequently, we can identify the limits in (3.3) and in (3.4) and conclude that, for almost every $x \in \Omega$,

$$\langle A_H(x)\nabla u_0(x), \nabla u_0(x)\rangle + \langle A_H(x)\nabla v_0(x), \nabla v_0(x)\rangle + 2\alpha\langle \nabla u_0(x), R\nabla v_0(x)\rangle = \\ = \int_{[\alpha,\beta]\times\mathbb{R}^2\times\mathbb{R}^2} \big(\langle a\xi,\xi\rangle + \langle a\eta,\eta\rangle + 2\alpha\langle \xi, R\eta\rangle\big)\, d\theta_x(a,\xi,\eta)$$

We are now in a position to apply the key Lemma 4.1 with the probability measure θ_x. Note that

$$\int_{[\alpha,\beta]\times\mathbb{R}^2\times\mathbb{R}^2} \xi\, d\theta_x(a,\xi,\eta) = \nabla u_0(x)\,, \quad \int_{[\alpha,\beta]\times\mathbb{R}^2\times\mathbb{R}^2} \eta\, d\theta_x(a,\xi,\eta) = \nabla v_0(x)$$

(this is the barycenter property, see Remark A.4). Note also that, since the projection of θ_x on $[\alpha,\beta]$ is equal to μ_x,

$$\int_{[\alpha,\beta]\times\mathbb{R}^2\times\mathbb{R}^2} \frac{2}{\alpha+a}\, d\theta_x(a,\xi,\eta) = \int_{[\alpha,\beta]} \frac{2}{a+\alpha}\, d\mu_x(a) = \frac{1}{a_*(x)}$$

Then Lemma 4.1 implies the inequality

$$\int_{[\alpha,\beta]\times\mathbb{R}^2\times\mathbb{R}^2} \big(\langle a\xi,\xi\rangle + \langle a\eta,\eta\rangle + 2\alpha\langle \xi, R\eta\rangle\big)\, d\theta_x(a,\xi,\eta) \geq a_*(x)\left|\nabla u_0(x) + R\nabla v_0(x)\right|^2,$$

valid for almost every $x \in \Omega$. The proof is now complete. □

Theorem 3.2. *Consider $\Omega \subset \mathbb{R}^2$ a two-dimensional domain and let (a_ε) be a bounded sequence of functions in $L^\infty(\Omega, [\alpha,\beta])$. Consider A_H the H-limit of $(a_\varepsilon I)$ and μ the Young measure associated to (a_ε). Suppose that $\mu_x(\{\alpha\}) > 0$ for almost every $x \in \Omega$ (in the sense of the Lebesgue measure). Then, for almost every $x \in \Omega$,*

$$\det\begin{bmatrix} \lambda_1(x) - a_*(x) & \alpha - a_*(x) \\ \alpha - a_*(x) & \lambda_2(x) - a_*(x) \end{bmatrix} \geq 0, \tag{3.5}$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are the eigenvalues of $A_H(x)$, while $a_*(x)$ is defined by

$$\frac{1}{a_*(x)} = \int_{[\alpha,\beta]} \frac{2}{a+\alpha}\, d\mu_x(a).$$

Note that we dropped out any regularity assumptions on the boundary of the domain Ω. The local character of the H-convergence and the local character of the Young measures allow us to do so: for any ball $\omega \subset \Omega$, the restriction of A_H to ω is still the H-limit of (a_ε) in ω, and the restriction of μ to $\omega \times [\alpha,\beta]$ is the Young measure associated to (a_ε) in ω (see Theorem 2.5 and Remark A.2 3)).

Proof of Theorem 3.2: Let U and V be two vectors in $\mathbb{R}^2$, arbitrarily fixed.

Let B be an arbitrary ball whose closure is included in Ω. Then there exists another ball ω such that $\overline{B} \subset \omega \subset \Omega$. Let u_0, v_0 be two functions in $\mathcal{C}^1(\omega)$ with compact supports and such that, for $x \in B$, $\nabla u_0(x) = U$ and $\nabla v_0(x) = V$. It is easy to build such functions; it suffices to consider a function $\varphi \in \mathcal{C}^1(\omega)$ with compact support and such that $\varphi = 1$ in B, and define $u_0(x) = \langle x, U\rangle\varphi(x)$, respectively $v_0(x) = \langle x, V\rangle\varphi(x)$. The functions u_0 and v_0 can be also viewed as functions in $\mathcal{C}^1(\Omega)$ with compact supports (by extension with zero).

Applying Proposition 3.1 with the domain ω, we deduce that, for almost every $x \in \omega$,

$$\begin{aligned}\langle A_H(x)\nabla u_0(x), \nabla u_0(x)\rangle + \langle A_H(x)\nabla v_0(x), \nabla v_0(x)\rangle +& \\ + 2\alpha\langle \nabla u_0(x), R\nabla v_0(x)\rangle \geq a_*(x)\left|\nabla u_0(x) + R\nabla v_0(x)\right|^2&.\end{aligned}$$

In particular, for almost every $x \in B$,

$$\langle A_H(x)U, U\rangle + \langle A_H(x)V, V\rangle + 2\alpha\langle U, RV\rangle \geq a_*(x)\left|U + RV\right|^2.$$

As the ball B was arbitrarily chosen and due to the local character of the H-convergence and of the Young measure, we conclude that, for fixed $U, V \in \mathbb{R}^2$ and for almost every $x \in \Omega$,

$$\langle A_H(x)U, U\rangle + \langle A_H(x)V, V\rangle + 2\alpha\langle U, RV\rangle \geq a_*(x)\left|U + RV\right|^2. \tag{3.6}$$

By a straightforward argument, one concludes the existence of a zero measure set $N \subset \Omega$ such that, for any $x \in \Omega \setminus N$ and for any $U, V \in \mathbb{R}^2$, the inequality (3.6) holds.

Now let us fix $x \in \Omega \setminus N$. By writing the vectors U and V as linear combinations between the two eigenvectors of $A_H(x)$ and by performing elementary computations, we verify that (3.6) is equivalent to the semi-positivity of the matrix

$$\begin{bmatrix} \lambda_1(x) - a_*(x) & \alpha - a_*(x) \\ \alpha - a_*(x) & \lambda_2(x) - a_*(x) \end{bmatrix}$$

By Proposition C.3, this matrix is semi-positive if and only if its determinant is non negative, which concludes the proof. □

The following result states the upper bound inequality for the two dimensional case. We give just a sketched proof, the techniques being similar in nature to the ones employed above for the lower bound.

Theorem 3.3. *Consider $\Omega \subset \mathbb{R}^2$ a two-dimensional domain and let (a_ε) be a bounded sequence of functions in $L^\infty(\Omega, [\alpha, \beta])$. Consider A_H the H-limit of $(a_\varepsilon I)$ and μ the Young measure associated to (a_ε). Suppose that $\mu_x(\{\beta\}) > 0$ for almost every $x \in \Omega$ (in the sense of the Lebesgue measure). Then, for almost every $x \in \Omega$,*

$$\det \begin{bmatrix} \lambda_1^{-1}(x) - a_{**}^{-1}(x) & \beta^{-1} - a_{**}^{-1}(x) \\ \beta^{-1} - a_{**}^{-1}(x) & \lambda_2^{-1}(x) - a_{**}^{-1}(x) \end{bmatrix} \geq 0 \tag{3.7}$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are the eigenvalues of $A_H(x)$, while $a_{**}(x)$ is defined by

$$a_{**}(x) = \int_{[\alpha,\beta]} \frac{2}{a^{-1} + \beta^{-1}} d\mu_x(a).$$

Sketch of proof: We begin by proving an auxiliary assertion similar to Proposition 3.1. Take $\Omega \subset \mathbb{R}^2$ a bounded domain with smooth boundary (of class $\mathcal{C}^q$ for some $q > 2$), and let u_0 and v_0 be two functions in $\mathcal{C}^1(\Omega)$ with compact support. Define the functions u_ε and v_ε as the solutions of the problems (3.1) and (3.2), respectively. Consider θ the Young measure associated to the triplet $(a_\varepsilon^{-1}, a_\varepsilon \nabla u_\varepsilon, a_\varepsilon \nabla v_\varepsilon)$. Note that the image of θ through the mapping $(b, \xi, \eta) \in [\beta^{-1}, \alpha^{-1}] \times \mathbb{R}^2 \times \mathbb{R}^2 \mapsto 1/b \in [\alpha, \beta]$ is μ. Using the same arguments as in Proposition 3.1, one can pass to the limit in the expression

$$\langle a_\varepsilon^{-1} a_\varepsilon \nabla u_\varepsilon, a_\varepsilon \nabla u_\varepsilon \rangle + \langle a_\varepsilon^{-1} a_\varepsilon \nabla v_\varepsilon, a_\varepsilon \nabla v_\varepsilon \rangle + 2\beta^{-1} \langle a_\varepsilon \nabla u_\varepsilon, R a_\varepsilon \nabla v_\varepsilon \rangle$$

and obtain as limit:

$$\int_{[\beta^{-1},\alpha^{-1}] \times \mathbb{R}^2 \times \mathbb{R}^2} \big(\langle b\xi, \xi \rangle + \langle b\eta, \eta \rangle + 2\beta^{-1} \langle \xi, R\eta \rangle \big) d\theta_x(b, \xi, \eta).$$

On the other hand, noting that $div(a_\varepsilon \nabla u_\varepsilon) = div(A_H \nabla u_0)$ and $rot(R a_\varepsilon \nabla v_\varepsilon) = div(a_\varepsilon \nabla v_\varepsilon)$ $= div(A_H \nabla v_0)$ are fixed functions in $L^2(\Omega)$, one can use the convergences (3.3) and identify the above limit as

$$\langle A_H(x) \nabla u_0(x), \nabla u_0(x) \rangle + \langle A_H(x) \nabla v_0(x), \nabla v_0(x) \rangle + 2\beta^{-1} \langle A_H \nabla u_0(x), R A_H \nabla v_0(x) \rangle.$$

We apply Lemma 4.1 and obtain, for almost every $x \in \Omega$, the inequality

$$\begin{aligned} &\langle A_H(x) \nabla u_0(x), \nabla u_0(x) \rangle + \langle A_H(x) \nabla v_0(x), \nabla v_0(x) \rangle + \\ &+ 2\beta^{-1} \langle A_H(x) \nabla u_0(x), R A_H(x) \nabla v_0(x) \rangle \geq a_{**}^{-1}(x) \left| A_H \nabla u_0(x) + R A_H \nabla v_0(x) \right|^2. \end{aligned}$$

Following the same arguments as in the proof of Theorem 3.2, we deduce that, for almost every $x \in \Omega$, for all vectors $U, V \in \mathbb{R}^2$,

$$\begin{aligned} \langle A_H(x) U, U \rangle + \langle A_H(x) V, V \rangle + 2\beta^{-1} \langle A_H(x) U, R A_H(x) V \rangle &\geq \\ \geq a_{**}^{-1}(x) \left| A_H(x) U + R A_H(x) V \right|^2&. \end{aligned}$$

For fixed x, we take $U' = A_H(x) U$ and $V' = A_H(x) V$ and re-write the above inequality as

$$\langle A_H^{-1}(x) U', U' \rangle + \langle A_H^{-1}(x) V', V' \rangle + 2\beta^{-1} \langle U', R V' \rangle \geq a_{**}^{-1}(x) \left| U' + R V' \right|^2.$$

Now, it suffices to write the vectors U' and V' as linear combinations of the two eigenvectors of $A_H(x)$ in order to conclude the proof. □

Remark 3.4. The bounds stated in Theorems 3.2 and 3.3 can be proven to be optimal under the same supplementary hypotheses as in Theorem 2 from [7]. Optimality of the lower and upper bounds above turns out by considering periodic sequences (a_ε) that give rise to matrices A_H and to Young measures μ that reach equality in (3.6) and (3.7), respectively.

The result below states the lower bound inequality for the n dimensional case ($n \geq 2$). We only point out the main steps of the proof.

Theorem 3.5. *Consider $\Omega \subset \mathbb{R}^n$ a domain and m an integer such that $1 \leq m \leq n$. Let (a_ε) be a bounded sequence of functions in $L^\infty(\Omega, [\alpha, \beta])$. Consider A_H the H-limit of $(a_\varepsilon I)$ and μ the Young measure associated to (a_ε). Suppose that $\mu_x(\{\alpha\}) > 0$ for almost every $x \in \Omega$ (in the sense of the Lebesgue measure). Then, for almost every $x \in \Omega$,*

$$\det \begin{bmatrix} \lambda_1(x) - a_*^{(m)}(x) & \alpha - a_*^{(m)}(x) & \dots & \alpha - a_*^{(m)}(x) \\ \alpha - a_*^{(m)}(x) & \lambda_2(x) - a_*^{(m)}(x) & \dots & \alpha - a_*^{(m)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha - a_*^{(m)}(x) & \alpha - a_*^{(m)}(x) & \dots & \lambda_m(x) - a_*^{(m)}(x) \end{bmatrix} \geq 0$$

where $\lambda_1(x), \lambda_2(x), \dots, \lambda_m(x)$ are any m eigenvalues of $A_H(x)$, while $a_*^{(m)}(x)$ is defined by

$$\frac{1}{a_*^{(m)}(x)} = \int_{[\alpha,\beta]} \frac{m}{a + (m-1)\alpha} d\mu_x(a). \tag{3.8}$$

Proof: We follow the steps in the proof of Theorem 3.2. We begin by using Lemma 4.3 in order to prove a generalization of Proposition 3.1: for any smooth domain $\Omega \subset \mathbb{R}^n$ and for any m functions $u_1, u_2, \dots, u_m$ of class C^1 with compact support in Ω, one has, almost everywhere in Ω,

$$\sum_k \langle A_H(x)\nabla u_k(x), \nabla u_k(x)\rangle + 2\alpha \sum_{k<l} \langle \nabla u_k(x), R_{kl} \nabla u_l(x)\rangle \geq$$
$$a_-(x) \sum_{k,i} u_{k,i}(x) u_{k,i}(x) + a_*^{(m)}(x) \Big(\sum_k u_{k,k}(x)\Big)^2 + a_*^{(2)}(x) \sum_{k \neq l} (u_{k,l}(x) - u_{l,k}(x)) u_{k,l}(x).$$

In the above sums and in what follows, the indices i, k and l run over the ranges $1 \leq k, l \leq m < i \leq n$. The matrices R_{kl} are rotations in $\mathbb{R}^n$; see Remark 4.2 in the next section. The functions a_- and $a_*^{(2)}$ are defined by $1/a_-(x) = \displaystyle\int_{[\alpha,\beta]} 1/a d\mu_x(a)$ and $1/a_*^{(2)}(x) = \displaystyle\int_{[\alpha,\beta]} 2/(a+\alpha) d\mu_x(a)$.

Using the local character of the H-convergence and the local character of Young measures, one proves that, for x outside a fixed negligible set $N \subset \Omega$ and for any m vectors $\eta_1, \eta_2, \dots, \eta_m \in \mathbb{R}^n$,

$$\sum_k \langle A_H(x)\eta_k, \eta_k\rangle + 2\alpha \sum_{k<l} (\eta_k^k \eta_l^l - \eta_k^l \eta_l^k) \geq$$
$$\geq a_-(x) \sum_{k,i} \eta_k^i \eta_k^i + a_*^{(m)}(x) \big(\sum_k \eta_k^k\big)^2 + a_*^{(2)}(x) \sum_{k \neq l} (\eta_k^l - \eta_l^k) \eta_k^l .$$

For each fixed x, the above inequality is valid in any coordinate system. We choose a base of eigenvectors of $A_H(x)$ and denote by $\lambda_i(x)$ the eigenvalues of $A_H(x)$ obtaining

$$\sum_{k,l} \lambda_l(x) \eta_k^l \eta_k^l + \sum_{k,i} \lambda_i(x) \eta_k^i \eta_k^i + \alpha \sum_{k \neq l} (\eta_k^k \eta_l^l - \eta_k^l \eta_l^k) \geq$$
$$\geq a_-(x) \sum_{k,i} \eta_k^i \eta_k^i + a_*^{(m)}(x) \big(\sum_k \eta_k^k\big)^2 + a_*^{(2)}(x) \sum_{k \neq l} (\eta_k^l - \eta_l^k) \eta_k^l .$$

By choosing the vectors η_k to be equal to the first m eigenvectors of $A_H(x)$, (which means that all coordinates η_k^i and η_k^l with $k \neq l$ are zero), we obtain

$$\sum_k \lambda_l(x) \eta_k^k \eta_k^k + \alpha \sum_{k \neq l} \eta_k^k \eta_l^l \geq a_*^{(m)}(x) \big(\sum_k \eta_k^k\big)^2 .$$

The above inequality can be rewritten as

$$\sum_k \eta_k^k \eta_k^k (\lambda_k(x) - a_*^{(m)}(x)) + \sum_{k \neq l} \eta_k^k \eta_l^l (\alpha - a_*^{(m)}(x)) \geq 0 \, .$$

This means that the matrix

$$\begin{bmatrix} \lambda_1(x) - a_*^{(m)}(x) & \alpha - a_*^{(m)}(x) & \dots & \alpha - a_*^{(m)}(x) \\ \alpha - a_*^{(m)}(x) & \lambda_2(x) - a_*^{(m)}(x) & \dots & \alpha - a_*^{(m)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha - a_*^{(m)}(x) & \alpha - a_*^{(m)}(x) & \dots & \lambda_m(x) - a_*^{(m)}(x) \end{bmatrix}$$

is semi-positive. According to Proposition C.3, this is equivalent to the non-negativity of the determinant of the matrix. $\square$

The generalization of Theorem 3.3 to more than two dimensions is not straightforward. For the lower bound, rotation operators were used which transform any gradient into a divergence-free vector field. For $n > 2$, there is no standard operation that transforms a divergence-free vector field into a gradient. So, the upper bound for the n dimensional case remains an open problem.

4. Key lemmas

The present section is dedicated to some results essential for the proofs of the theorems in Section 3. These results combine fine measure theory and minimization of quadratic forms on affine spaces. Recall that R is a rotation operator in $\mathbb{R}^2$ ($R^2 = -I$, $R^T = -R$).

Lemma 4.1. *Consider a probability measure θ on $[\alpha, \beta] \times \mathbb{R}^2 \times \mathbb{R}^2$ with the properties $\theta(\{\alpha\} \times \mathbb{R}^2 \times \mathbb{R}^2) > 0$ and $\int_{[\alpha,\beta]\times\mathbb{R}^2\times\mathbb{R}^2} (|\xi|^2 + |\eta|^2) d\theta(\lambda, \xi, \eta) < +\infty$. Let $\varphi : [\alpha, \beta] \times \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by $\varphi(\lambda, \xi, \eta) = \langle \lambda\xi, \xi\rangle + \langle \lambda\eta, \eta\rangle + 2\alpha\langle \xi, R\eta\rangle$.*
Denote by $\lambda_* \in [\alpha, \beta]$, $\xi_0 \in \mathbb{R}^2$ and $\eta_0 \in \mathbb{R}^2$ the following quantities:

$$\frac{1}{\lambda_*} = \int_{[\alpha,\beta]\times\mathbb{R}^2\times\mathbb{R}^2} \frac{2}{\alpha + \lambda} \, d\theta(\lambda, \xi, \eta), \tag{4.1}$$

$$\xi_0 = \int_{[\alpha,\beta]\times\mathbb{R}^2\times\mathbb{R}^2} \xi \, d\theta(\lambda, \xi, \eta), \tag{4.2}$$

$$\eta_0 = \int_{[\alpha,\beta]\times\mathbb{R}^2\times\mathbb{R}^2} \eta \, d\theta(\lambda, \xi, \eta). \tag{4.3}$$

Then the following estimation holds

$$\int_{[\alpha,\beta]\times\mathbb{R}^2\times\mathbb{R}^2} \varphi(\lambda, \xi, \eta) d\theta(\lambda, \xi, \eta) \geq \lambda_* |\xi_0 + R\eta_0|^2. \tag{4.4}$$

Proof: By the representation result (Proposition A.5) and the remark following it, there exists a function $f : [0, 1] \to [\alpha, \beta] \times \mathbb{R}^2 \times \mathbb{R}^2$ which transports the Lebesgue measure from $[0, 1]$ into the probability θ, that is, $\theta(B) = \mathcal{L}_1(f^{-1}(B))$ for all $B \in \mathcal{B}([\alpha, \beta] \times \mathbb{R}^2 \times \mathbb{R}^2)$. Consider the components of $f = (a_\theta, g_\theta, h_\theta)$ as $a_\theta : [0, 1] \to [\alpha, \beta]$ and $g_\theta, h_\theta : [0, 1] \to \mathbb{R}^2$. The property $\int_{[\alpha,\beta]\times\mathbb{R}^2\times\mathbb{R}^2} (|\xi|^2 + |\eta|^2) d\theta(\lambda, \xi, \eta) < +\infty$ ensures that $(g_\theta, h_\theta) \in L^2([0, 1]; \mathbb{R}^2 \times \mathbb{R}^2)$. The definition of f implies, for any continuous function $\sigma : [\alpha, \beta] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, the identity

$$\int_{[\alpha,\beta]\times\mathbb{R}^2\times\mathbb{R}^2} \sigma(\lambda, \xi, \eta) \, d\theta(\lambda, \xi, \eta) = \int_{[0,1]} \sigma(a_\theta(x), g_\theta(x), h_\theta(x)) dx.$$

Then

$$\frac{1}{\lambda_*} = \int \frac{2}{\alpha + \lambda} \, d\theta(\lambda, \xi, \eta) = \int_{[0,1]} \frac{2}{\alpha + a_\theta(x)} \, dx \tag{4.5}$$

Consider the following affine space

$$\mathcal{X} = \{(g, h) \in L^2([0, 1]; \mathbb{R}^2 \times \mathbb{R}^2) | \int_{[0,1]} g(x) dx = \xi_0, \int_{[0,1]} h(x) dx = \eta_0\}. \tag{4.6}$$

Let $\Phi : \mathcal{X} \mapsto \mathbb{R}$ be the functional defined by

$$\Phi(g,h) = \int_{[0,1]} [\langle a_\theta(x)g(x), g(x)\rangle + \langle a_\theta(x)h(x), h(x)\rangle + 2\alpha\langle g(x), Rh(x)\rangle]dx. \tag{4.7}$$

Then Φ is a quadratic form on $\mathcal{X}$ (see Appendix C for general properties of quadratic forms). Let Ψ be its associated bi-affine functional (on $\mathcal{X} \times \mathcal{X}$) and $\delta\delta\Psi$ be the corresponding bilinear functional (on $T\mathcal{X} \times T\mathcal{X}$, where $T\mathcal{X}$ is the linear space tangent to $\mathcal{X}$). Then $\delta\delta\Psi((\delta g, \delta h), (\delta g, \delta h)) \geq 0$, $\forall (\delta g, \delta h) \in T\mathcal{X}$. Indeed, $a_\theta(x) \geq \alpha$ for all $x \in [0,1]$ and therefore

$$\begin{aligned}
\delta\delta\Psi((\delta g,&\delta h), (\delta g, \delta h)) = \\
&= \int_{[0,1]} [\langle a_\theta(x)\delta g(x), \delta g(x)\rangle + \langle a_\theta(x)\delta h(x), \delta h(x)\rangle + 2\alpha\langle \delta g(x), R\delta h(x)\rangle]dx \geq \\
&\geq \int_{[0,1]} [\alpha\langle \delta g(x), \delta g(x)\rangle + \alpha\langle \delta h(x), \delta h(x)\rangle + 2\alpha\langle \delta g(x), R\delta h(x)\rangle]dx = \\
&= \int_{[0,1]} \langle \delta g(x) + R\delta h(x), \delta g(x) + R\delta h(x)\rangle dx \geq 0\,.
\end{aligned}$$

Thus, by Proposition C.2, it turns out that a necessary and sufficient condition for a point $(g,h) \in \mathcal{X}$ to be a minimum for Φ is $\delta\Psi((g,h),(\delta g, \delta h)) = 0$, $\forall\,(\delta g, \delta h) \in T\mathcal{X}$. A straightforward calculation gives

$$\begin{aligned}
\delta\Psi((g,h),&(\delta g, \delta h)) = \\
&= \int_{[0,1]} [2\langle a_\theta(x)g(x), \delta g(x)\rangle + 2\langle a_\theta(x)h(x), \delta h(x)\rangle \\
&\qquad\qquad + 2\alpha\langle \delta g(x), Rh(x)\rangle + 2\alpha\langle g(x), R\delta h(x)\rangle]dx = \\
&= 2\int_{[0,1]} [\langle a_\theta(x)g(x) + \alpha Rh(x), \delta g(x)\rangle + \langle a_\theta(x)h(x) + \alpha R^T g(x), \delta h(x)\rangle]dx.
\end{aligned} \tag{4.8}$$

Recall that $T\mathcal{X}$, the tangent space of $\mathcal{X}$, is the set of vector fields $(\delta g, \delta h)$ such that $\int_{[0,1]} \delta g(x)dx = 0$ and $\int_{[0,1]} \delta h(x)dx = 0$. A necessary and sufficient condition of minimum writes then

$$\begin{aligned}
&\int_{[0,1]} [\langle a_\theta(x)g(x) + \alpha Rh(x), \delta g(x)\rangle + \langle a_\theta(x)h(x) + \alpha R^T g(x), \delta h(x)\rangle]dx = 0\,, \\
&\qquad\qquad \text{for all } \delta g \text{ and } \delta h \text{ such that } \int_{[0,1]} \delta g(x)dx = 0 \text{ and } \int_{[0,1]} \delta h(x)dx = 0.
\end{aligned} \tag{4.9}$$

We deduce the following condition of minimum (see Proposition C.4):

$$\begin{aligned}
a_\theta(x)g(x) + \alpha Rh(x) &= \mathit{constant} \in \mathbb{R}^2, \\
a_\theta(x)h(x) + \alpha R^T g(x) &= \mathit{constant} \in \mathbb{R}^2.
\end{aligned}$$

Let c and d be two vectors in $\mathbb{R}^2$ such that

$$\begin{cases} a_\theta(x)g(x) + \alpha Rh(x) = c, & \text{for a. e. } x \in [0,1], \\ a_\theta(x)h(x) - \alpha Rg(x) = d, & \text{for a. e. } x \in [0,1]. \end{cases} \tag{4.10}$$

The quadratic form Φ may be written as follows:

$$\Phi(g,h) = \int_{[0,1]} [\langle a_\theta(x)g(x) + \alpha Rh(x), g(x)\rangle + \langle a_\theta(x)h(x) - \alpha Rg(x), h(x)\rangle] dx.$$

Suppose that there exists $(g_0, h_0) \in \mathcal{X}$ satisfying (4.10) (we shall prove the existence later). Then

$$\Phi(g_0, h_0) = \int_{[0,1]} [\langle c, g_0(x)\rangle + \langle d, h_0(x)\rangle] dx = \langle c, \xi_0\rangle + \langle d, \eta_0\rangle. \tag{4.11}$$

In the sequel we shall compute the vectors c and d which will allow us to deduce the minimum value of Φ. The property $\theta(\{\alpha\} \times \mathbb{R}^2 \times \mathbb{R}^2) > 0$ implies that on a set of positive measure $a_\theta(x) = \alpha$. Let us then consider a point x_0 such that:

$$\begin{cases} a_\theta(x_0) &= \alpha, \\ a_\theta(x_0)g(x_0) + \alpha Rh(x_0) &= c, \\ a_\theta(x_0)Rh(x_0) + \alpha g(x_0) &= Rd. \end{cases}$$

Then, replacing $a_\theta(x_0)$ by α in the considered point x_0, we have

$$\begin{cases} \alpha g(x_0) + \alpha Rh(x_0) &= c, \\ \alpha Rh(x_0) + \alpha g(x_0) &= Rd \end{cases}$$

and consequently

$$c = Rd. \tag{4.12}$$

From system (4.10) and having in mind (4.12) it turns out that $(a_\theta(x)+\alpha)g_0(x) + (a_\theta(x)+\alpha)Rh_0(x) = 2c$ for almost every x in $[0,1]$, consequently

$$g_0(x) + Rh_0(x) = \frac{2c}{a_\theta(x) + \alpha}.$$

By integration in x we obtain

$$\xi_0 + R\eta_0 = c \int_{[0,1]} \frac{2}{a_\theta(x) + \alpha} dx = \frac{c}{\lambda_*}$$

and we get the value of c

$$c = \lambda_*(\xi_0 + R\eta_0). \tag{4.13}$$

Consequently we are able to compute the minimum value of Φ, by replacing c and d in (4.11) as follows :

$$\begin{aligned}\Phi(g_0,h_0) &= \lambda_*\langle \xi_0 + R\eta_0, \xi_0\rangle + \lambda_*\langle \eta_0 - R\xi_0, \eta_0\rangle \\ &= \lambda_*(\langle \xi_0,\xi_0\rangle + \langle \eta_0,\eta_0\rangle + 2\langle \xi_0, R\eta_0\rangle) = \lambda_*|\xi_0 + R\eta_0|^2.\end{aligned}$$

Let us prove the existence of the critical point $(g_0,h_0)\in\mathcal{X}$. It suffices to build a pair of functions $(g_0,h_0)\in\mathcal{X}$ satisfying conditions (4.10) that can be written in the form below, for almost every x in $[0,1]$:

$$\begin{cases} a_\theta(x)g(x) + \alpha R h(x) = c, \\ \alpha R g(x) - a_\theta(x)h(x) = Rc. \end{cases} \tag{4.14}$$

Consider $x\in[0,1]$ such that $a_\theta(x)>\alpha$. Adding the previous two identities after multiplying the first one by a_θ and the second one by αR, we obtain $(a_\theta^2(x)-\alpha^2)g(x) = (a_\theta(x)-\alpha)c$ and hence

$$g_0(x) = \frac{c}{a_\theta(x)+\alpha}.$$

On the other hand, from system (4.14) and having in mind the above expression of $g_0(x)$, we obtain

$$h_0(x) = \frac{-Rc}{a_\theta(x)+\alpha}.$$

Using (4.13) we get the following expressions for $g_0(x)$ and $h_0(x)$:

$$\begin{aligned} g_0(x) &= \frac{\lambda_*(\xi_0 + R\eta_0)}{a_\theta(x)+\alpha}, \\ h_0(x) &= \frac{\lambda_*(\eta_0 - R\xi_0)}{a_\theta(x)+\alpha}, \end{aligned}$$

which hold in almost every point x in $[0,1]$ where $a_\theta(x)>\alpha$.

Now it remains to define the functions g_0 and h_0 in the points x where $a_\theta(x)=\alpha$. This is done having in mind the definition of the space $\mathcal{X}$ where (g_0,h_0) belongs and conditions (4.14) that in this case degenerate in an unique condition.

For any x in the set $\{x|\, a_\theta(x)=\alpha\}$ consider

$$g_0(x) = v, \quad h_0(x) = w,$$

where v and w are constant vectors to be determined.

Indeed, from the condition $\displaystyle\int_{[0,1]} g_0(x)dx = \xi_0$ (definition of $\mathcal{X}$) it turns out that

$$v = \left[\xi_0 - \lambda_*(\xi_0 + R\eta_0)\int_{a_\theta>\alpha}\frac{dx}{a_\theta(x)+\alpha}\right] / \mathcal{L}_1(\{x|\, a_\theta(x)=\alpha\}).$$

From the condition $\displaystyle\int_{[0,1]} h_0(x)dx = \eta_0$ in the definition of $\mathcal{X}$, we obtain

$$w = \left[\eta_0 - \lambda_*(\eta_0 - R\xi_0)\int_{a_\theta>\alpha} \frac{dx}{a_\theta(x)+\alpha}\right] / \mathcal{L}_1(\{x|\, a_\theta(x) = \alpha\}).$$

In order to complete the proof we still have to verify that in the set $\{x|\, a_\theta(x) = \alpha\}$ (where the two equations in (4.14) coincide), the following equality holds:

$$a_\theta(x)g(x) + \alpha Rh(x) = \lambda_*(\xi_0 + R\eta_0),$$

which can be done by straightforward calculations. $\square$

We shall prove a generalization of Lemma 4.1 for vectors in $\mathbb{R}^n$.

Remark 4.2. As there is no standard rotation operator in $\mathbb{R}^n$, we shall use several "rotation" operators R_{kl}, defined as follows. Consider $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m$ a number of orthonormal vectors in $\mathbb{R}^n$. For $1 \le k, l \le m$, $k \ne l$, define the linear mapping $R_{kl} : \mathbb{R}^n \to \mathbb{R}^n$ as the orthogonal projection onto the plane defined by $\vec{e}_k$ and $\vec{e}_l$, followed by the usual rotation in that plane. In coordinate notation, assuming that the vectors $\vec{e}_k$ are the first m vectors in the canonical basis of $\mathbb{R}^n$, and supposing $k < l$:

$$R_{kl}(\xi^1, \xi^2, \ldots, \xi^k, \ldots, \xi^l, \ldots, \xi^n) = (0, 0, \ldots, \xi^l, \ldots, -\xi^k, \ldots, 0)$$

For instance, in $\mathbb{R}^3$, the operators R_{kl} act like this: $R_{12}(\xi^1, \xi^2, \xi^3) = (\xi^2, -\xi^1, 0)$, $R_{13}(\xi^1, \xi^2, \xi^3) = (\xi^3, 0, -\xi^1)$, $R_{23}(\xi^1, \xi^2, \xi^3) = (0, \xi^3, -\xi^2)$. Note that $R_{lk} = -R_{kl}$, $R_{kl}^T = -R_{kl}$, but the operators R_{kl} are not invertible (unlike the rotation operator in $\mathbb{R}^2$).

In Lemma 4.3 below the notations are different from Lemma 4.1. The variable vectors in $\mathbb{R}^n$ are designated by ξ_k while η_k are their arithmetic means. Also, the functions given by the representation theorem are designated by $h_k(x)$ while $g_k(x)$ are variable vector fields in $L^2([0,1]; \mathbb{R}^n)$.

Lemma 4.3. *Let n and m be two integers such that $2 \le m \le n$. Consider a probability measure θ on $[\alpha, \beta] \times (\mathbb{R}^n)^m$ with the properties $\theta(\{\alpha\} \times (\mathbb{R}^n)^m) > 0$ and $\displaystyle\int_{[\alpha,\beta]\times(\mathbb{R}^n)^m} |\xi_k|^2 d\theta(\lambda, \xi_1, \xi_2 \ldots, \xi_m) < +\infty$. Let $\varphi : [\alpha, \beta] \times (\mathbb{R}^n)^m \mapsto \mathbb{R}$ be defined by $\varphi(\lambda, \xi_1, \xi_2 \ldots, \xi_m) = \sum_{k=1}^m \langle \lambda\xi_k, \xi_k\rangle + 2\alpha \sum_{1\le k<l\le m} \langle \xi_k, R_{kl}\xi_l\rangle$. In coordinate notation, $\varphi(\lambda, \xi_1, \xi_2 \ldots, \xi_m) = \sum_{k=1}^m \big(\lambda \sum_{i=1}^n \xi_k^i \xi_k^i\big) + 2\alpha \sum_{1\le k<l\le m} (\xi_k^k \xi_l^l - \xi_k^l \xi_l^k)$, where $\xi_k = (\xi_k^1, \xi_k^2, \ldots, \xi_k^n)$. Then*

$$\int_{[\alpha,\beta]\times(\mathbb{R}^n)^m} \varphi(\lambda, \xi_1, \xi_2 \ldots, \xi_m) d\theta(\lambda, \xi_1, \xi_2 \ldots, \xi_m) \ge$$
$$\ge \lambda_- \sum_{k,i} \eta_k^i \eta_k^i + \lambda_*^{(m)} \Big(\sum_k \eta_k^k\Big)^2 + \lambda_*^{(2)} \sum_{l\ne k} (\eta_k^l - \eta_l^k)\eta_k^l\,,$$

where the indices i, k and l of the sums run over the ranges: $1 \leq k, l \leq m < i \leq n$. In the above formula, the vectors $\eta_1, \eta_2, \ldots, \eta_m$ represent the barycenters of the different projections of θ:

$$\eta_k = \int_{[\alpha,\beta]\times(\mathbb{R}^n)^m} \xi_k \, d\theta\,(\lambda, \xi_1, \xi_2 \ldots, \xi_m),$$

while $\lambda_*^{(m)}$ is defined as

$$\frac{1}{\lambda_*^{(m)}} = \int_{[\alpha,\beta]\times(\mathbb{R}^n)^m} \frac{m}{\lambda + (m-1)\alpha} \, d\theta\,(\lambda, \xi_1, \xi_2 \ldots, \xi_m).$$

Note that

$$\frac{1}{\lambda_*^{(2)}} = \int_{[\alpha,\beta]\times(\mathbb{R}^n)^m} \frac{2}{\lambda + \alpha} \, d\theta\,(\lambda, \xi_1, \xi_2 \ldots, \xi_m).$$

We have used the symbol λ_- in order to denote the harmonic mean $\lambda_*^{(1)}$ because it is the usual notation in homogenization theory:

$$\frac{1}{\lambda_-} = \frac{1}{\lambda_*^{(1)}} = \int_{[\alpha,\beta]\times(\mathbb{R}^n)^m} \frac{1}{\lambda} \, d\theta\,(\lambda, \xi_1, \xi_2 \ldots, \xi_m).$$

Proof: We shall follow the lines of the proof of Lemma 4.1. First, we use the representation result (Proposition A.5) in order to transport the integrals on the probability space $[0,1]$. Namely, there exists a measurable function $f : [0,1] \to [\alpha, \beta] \times (\mathbb{R}^n)^m$ whose components are $f = (a, h_1, h_2, \ldots, h_m)$, verifying

$$\int_{[\alpha,\beta]\times(\mathbb{R}^n)^m} \varphi(\lambda, \xi_1, \xi_2 \ldots, \xi_m) \, d\theta(\lambda, \xi_1, \xi_2 \ldots, \xi_m) = \int_{[0,1]} \varphi(a(x), h_1(x), h_2(x), \ldots, h_m(x)) dx.$$

We consider the following affine space

$$\mathcal{X} = \{(g_1, g_2, \ldots, g_m) \in L^2\big([0,1]; (\mathbb{R}^n)^m\big) | \int_{[0,1]} g_l(x) dx = \eta_l \,, l = 1, 2, \ldots, m\}.$$

We shall reserve the letter h for the vector fields given by the representation theorem, and use the letter g for vector fields running in the space $\mathcal{X}$. The function a (the first component of f) remains fixed.

Define the quadratic form $\Phi : \mathcal{X} \to \mathbb{R}$ by

$$\Phi(g_1, g_2, \ldots, g_m) = \int_{[0,1]} \left[\sum_{l=1}^{m} \langle a(x)\, g_l(x), g_l(x) \rangle + 2\alpha \sum_{1 \leq k \leq l \leq m} \langle g_k(x), R_{kl} g_l(x) \rangle \right].$$

Note that

$$\int_{[\alpha,\beta]\times(\mathbb{R}^n)^m} \varphi(\lambda, \xi_1, \xi_2, \ldots, \xi_m) d\theta(\lambda, \xi_1, \xi_2, \ldots, \xi_m) = \Phi(h_1, h_2, \ldots, h_m)\,.$$

The bilinear functional $\delta\delta\Psi$ (see Appendix C) has the non-negativity property:

$$\begin{aligned}
&\delta\delta\Psi((\delta g_1,\delta g_2,\ldots,\delta g_m),(\delta g_1,\delta g_2,\ldots,\delta g_m)) = \\
&\quad = \int_{[0,1]} \left[a \sum_{l=1}^{m}\sum_{i=1}^{n} \delta g_l^i \delta g_l^i + 2\alpha \sum_{1\le k<l\le m} (\delta g_k^k \delta g_l^l - \delta g_k^l \delta g_l^k) \right] dx \ge \\
&\quad \ge \int_{[0,1]} \left[\alpha \sum_{l=1}^{m}\sum_{i=1}^{n} \delta g_l^i \delta g_l^i + 2\alpha \sum_{1\le k<l\le m} (\delta g_k^k \delta g_l^l - \delta g_k^l \delta g_l^k) \right] dx \ge \\
&\quad \ge \int_{[0,1]} \alpha \left[\sum_{l=1}^{m}\sum_{k=1}^{m} \delta g_l^k \delta g_l^k + 2 \sum_{1\le k<l\le m} (\delta g_k^k \delta g_l^l - \delta g_k^l \delta g_l^k) \right] dx = \\
&\quad = \alpha \int_{[0,1]} \left[\Big(\sum_{l=1}^{m} \delta g_l^l \Big)^2 + \sum_{1\le k<l\le m} \big(\delta g_k^l - \delta g_l^k\big)^2 \right] dx \ge 0
\end{aligned}$$

By making some calculations and by employing Proposition C.4, we show that the minimum condition $\delta\Psi((g_1,g_2,\ldots,g_m),(\delta g_1,\delta g_2,\ldots,\delta g_m)) = 0$, $\forall\,(\delta g_1,\delta g_2,\ldots,\delta g_m) \in T\mathcal{X}$ given in Proposition C.2 is equivalent to the following three conditions

$$a(x) g_l^i(x) = c_l^i\,, \quad 1 \le l \le m < i \le n \tag{4.15}$$

$$(a(x)-\alpha) g_l^l(x) + \alpha \sum_{1\le k\le m} g_k^k = c_l^l\,, \quad 1 \le l \le m \tag{4.16}$$

$$a(x) g_k^l(x) - \alpha g_l^k(x) = c_k^l\,, \quad 1 \le k,l \le m\,,\ k \ne l \tag{4.17}$$

for some real constants c_l^i, c_l^l, c_k^l. In the sequel we compute these constants which will allow us to deduce the minimum value of Φ.

By dividing equation (4.15) by $a(x)$ and integrating, we conclude that $c_l^i = \lambda_- \eta_l^i$ for $1 \le l \le m < i \le n$.

By taking in equation (4.17) some $x_0 \in [0,1]$ such that $a(x_0) = \alpha$ and by switching k and l we conclude that $c_k^l = -c_k^l$ for $1 \le k,l \le m$, $k \ne l$. Then, by substracting one equation from the other, dividing by $a(x)+\alpha$ and integrating, we get $c_k^l = (\eta_k^l - \eta_l^k)\lambda_*^{(2)}$.

By taking in equation (4.16) some $x_0 \in [0,1]$ such that $a(x_0) = \alpha$ we deduce that $\alpha \sum_{k=1}^{m} g_k^k(x_0) = c_l^l$, which means that all constants c_l^l are equal (for $1 \le l \le m$). We shall denote their common value by c.

By adding equations (4.16) (with $1 \le k \le m$), we obtain $(a(x)+(m-1)\alpha)\sum_{l=1}^{m} g_l^l(x) = mc$. We divide this equality by $a(x)+(m-1)\alpha$ and integrate in order to obtain $c = \lambda_*^{(m)} \sum_{l=1}^{m} \eta_l^l$.

For vector fields g_k satisfying (4.15)–(4.17), $\Phi(g_1,g_2,\ldots,g_m)$ is the minimum value of Φ. In the sequel, we compute this value. The indices i, k and l of the sums below run over the ranges: $m < i \le n$, $1 \le k, l \le m$.

$$\Phi(g_1, g_2, \ldots, g_m) = \int_{[0,1]} \Big[\sum_{k,i} a g_k^i g_k^i + \sum_k (a g_k^k + \alpha \sum_{l \neq k} g_l^l) g_k^k + \sum_{l \neq k} (a g_k^l - \alpha g_l^k) g_k^l \Big] dx =$$

$$= \int_{[0,1]} \Big[\sum_{k,i} c_k^i g_k^i + c \sum_k g_k^k + \sum_{l \neq k} c_k^l g_k^l \Big] dx =$$

$$= \lambda_- \sum_{k,i} \eta_k^i \eta_k^i + \lambda_*^{(m)} \sum_l \eta_l^l \sum_k \eta_k^k + \lambda_*^{(2)} \sum_{l \neq k} (\eta_k^l - \eta_l^k) \eta_k^l$$

The proof of the existence of vector fields g_k satisfying equations (4.15)–(4.17) is analogous to the one presented for Lemma 4.1. $\square$

Appendix A. **Young measures**

In the sequel we make a brief recall on Young measures. For more details see e.g. E. Balder [5], J. Ball [6], M. Valadier [33].

Let Ω be a bounded domain in $\mathbb{R}^n$, denote by $\mathcal{L}_n$ the Lebesgue measure on $\mathbb{R}^n$ and by $\mathcal{B}(\Omega)$ the Borel σ-field of Ω.

We call *Young measure* any positive measure μ on $\Omega \times S$ (S is a metrizable space) whose projection on Ω is $\mathcal{L}_n$. Let $\mathcal{Y}(\Omega \times S)$ be the set of all Young measures on $\Omega \times S$.

We will not distinguish μ from its disintegration $(\mu_x)_{x \in \Omega}$ which is a measurable family of probabilities on S such that for any $\psi : \Omega \times S \to \mathbb{R}$, μ-integrable,

$$\int_{\Omega \times S} \psi \, d\mu = \int_\Omega \int_S \psi(x, \xi) \, d\mu_x(\xi) dx.$$

To each measurable function $a : \Omega \to S$ we associate a Young measure μ_a, with support in the graph of a, defined by

$$\langle \mu_a, \phi \rangle = \int_{\Omega \times S} \phi(x, \lambda) d\mu_a(x, \lambda) = \int_\Omega \phi(x, a(x)) dx,$$

for all positive $\phi : \Omega \times S \to \mathbb{R}$ that are measurable.

On $\mathcal{Y}(\Omega \times S)$ we consider the *narrow* topology, i. e., the weakest topology that makes continuous the maps

$$\mu \mapsto \int_{\Omega \times S} \phi(x, \lambda) d\mu(x, \lambda),$$

for all bounded Carathéodory integrands ϕ.

Remark A.1. 1) The set of Young measures associated to a sequence of functions uniformly bounded in $L^1(\Omega; \mathbb{R}^d)$ is relatively compact in $\mathcal{Y}(\Omega \times \mathbb{R}^d)$.

2) If S is a metrizable compact space, then every set of Young measures associated to a sequence of measurable functions $a_\varepsilon : \Omega \to S$ is relatively compact in $\mathcal{Y}(\Omega \times S)$.

Remark A.2. 1) Given a uniformly bounded sequence (a_ε) in $L^1(\Omega;\mathbb{R}^d)$, using Remark A.1 1),we may assume that, up to a subsequence of (a_ε), the sequence of their associated Young measures, narrow converges to some $\mu = (\mu_x)_{x\in\Omega} \in \mathcal{Y}(\Omega \times \mathbb{R}^d)$.
2) For a sequence of functions $a_\varepsilon : \Omega \to S$ we say that μ is the Young measure associated to the sequence (a_ε), or, that (a_ε) gives rise to the Young measure μ, if the Young measures associated to (a_ε) narrow converge to μ, i. e., for all bounded Carathéodory integrand ϕ,

$$\int_\Omega \phi(x, a_\varepsilon(x))\ dx \to \int_{\Omega\times S} \phi(x,\lambda)\ d\mu(x,\lambda).$$

3) The Young measure associated to a sequence has local character: Consider $a_\varepsilon : \omega_1 \to S$ and $b_\varepsilon : \omega_2 \to S$ two sequences of measurable functions giving rise to the Young measures μ_1 and μ_2, respectively. Suppose that $\omega_1 \cap \omega_2 \neq \emptyset$. If $a_\varepsilon(x) = b_\varepsilon(x)$ for all $\varepsilon > 0$ and for almost every $x \in \omega_1 \cap \omega_2$, then the Young measures μ_1 and μ_2 coincide on $(\omega_1 \cap \omega_2) \times S$. More precisely, $\mu_1(A \times B) = \mu_2(A \times B)$ for any Lebesgue measurable set $A \subset \omega_1 \cap \omega_2$ and any $B \in \mathcal{B}(S)$.

The following result, known in the literature as the fundamental theorem for Young measures (see [33], Section 4, and [6]), allows us to consider a larger class of admissible functions with respect to the narrow convergence, provided some uniform integrability condition is satisfied. A sequence (u_ε) in $L^1(\Omega;\mathbb{R}^d)$ is said to be *uniformly integrable* if (u_ε) is bounded in $L^1(\Omega;\mathbb{R}^d)$ and if for all $\eta > 0$ there exists $\delta > 0$ such that for any measurable subset A of Ω verifying $\mathcal{L}_n(A) < \delta$, the estimate $\sup\limits_\varepsilon \int_A \|u_\varepsilon(x)\|dx < \eta$ holds.

Theorem A.3. *Let S be a metrizable, locally compact, separable, complete space. Let $a_\varepsilon : \Omega \to S$ be a sequence of measurable functions and let μ be the Young measure associated to (a_ε), in the sense of Remark A.2 2). If $\psi : \Omega \times S \to \mathbb{R}$ is a Carathéodory integrand such that $(\psi(\cdot, a_\varepsilon(\cdot))$ is uniformly integrable in Ω, then*

$$\int_{\Omega\times S} \psi(x,\lambda)d\mu(x,\lambda) = \lim_\varepsilon \int_\Omega \psi\left(x, a_\varepsilon(x)\right) dx. \tag{A.1}$$

Remark A.4. If the sequence (a_ε) converges weakly in $L^1(\Omega)$ to a function $a \in L^1(\Omega)$, then the weak limit a is the barycenter of the Young measure μ associated to (a_ε) :

$$a(x) = \int_S \lambda d\mu_x(\lambda), \text{ a. e. } x \in \Omega.$$

Although it is not directly linked to Young measures, we include in this appendix a representation result.

Proposition A.5. *Consider θ a given probability measure on $\mathbb{R}^n$. Then there exists a measurable function $f : [0,1] \mapsto \mathbb{R}^n$ which transports the Lebesgue measure of $[0,1]$ into θ, that is,*

$$\theta(B) = \mathcal{L}_n(f^{-1}(B))$$

for all borelian sets $B \in \mathcal{B}(\mathbb{R}^n)$.

Proof: Note that for $n = 1$ one can take for instance the nonincreasing function given by

$$f(x) := \sup\{t \in \mathbb{R} : \theta([t, +\infty]) > x\}$$

(this construction is usual in rearrangement techniques, see [16]).
For $n > 1$, we use a result of probability theory (see [21], Theorem 6.4.3) which says that there exists an isomorphism of probability spaces between $(\mathbb{R}^n, \theta)$ and $([0,1], p)$, where p has the form $p = \gamma + (1 - \|\gamma\|)\mathcal{L}_1$. Here, γ is a finite or countable sum of Dirac masses: $\gamma = \sum c_k \delta_{\xi_k}$, satisfying $\|\gamma\| = \sum c_k \leq 1$ and $\mathcal{L}_1$ is the one-dimensional Lebesgue measure. Now it suffices to use the result for the probability p (with $n = 1$) in order to conclude the proof. $\square$

Remark A.6. The above result holds for probability measures θ defined on a finite product of subintervals of $\mathbb{R}$. In this case the function f takes values in the same product of intervals.

Appendix B. **Regularity result**

In this Appendix we state a result due to N. Meyers [23].

Theorem B. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $\mathcal{C}^q$, with $q > 2$. For every $\vec{f} \in L^1(\Omega, \mathbb{R}^n)$, let $u \in H_0^1(\Omega)$ be the unique solution of the elliptic equation $div(A\nabla u) = div \vec{f}$, where the matrix function A is such that its components belong to $L^\infty(\Omega)$ and $\alpha I \leq A(x) \leq \beta I$, with $0 < \alpha < \beta$. There exists then an exponent $Q \leq q$ such that, for every p with $2 < p < Q$, if $\vec{f} \in L^p(\Omega, \mathbb{R}^n)$ then $\nabla u \in L^p(\Omega, \mathbb{R}^n)$ and $\|\nabla u\|_p \leq C\|\vec{f}\|_p$, where the constant C is independent of $\vec{f}$ (C depends only on the domain Ω, on the constants p, Q and on the ratio β/α).*

Appendix C. **Quadratic forms on affine spaces and other auxiliary results**

Let X be an affine space, and let TX be its associated linear space (TX is the tangent space to X in any point $x \in X$). Let $\ell : X \to \mathbb{R}$ be an affine functional. For each

$\vec{v} \in TX$, the expression $\ell(x+\vec{v}) - \ell(x)$ is independent of $x \in X$. We define the functional $\delta\ell(\vec{v}) = \ell(x+\vec{v}) - \ell(x)$, which turns out to be linear on TX.

Let $\Psi : X \times X \to \mathbb{R}$ a bi-affine functional (that is, an application affine in each of its two arguments). For $\vec{v} \in TX$, the expression $\Psi(x, y+\vec{v}) - \Psi(x,y)$ is independent of $y \in X$. We define the functional $\delta\Psi : X \times TX \to \mathbb{R}$ by $\delta\Psi(x,\vec{v}) = \Psi(x, y+\vec{v}) - \Psi(x,y)$, it turns out that $\delta\Psi$ is affine in the first argument and linear in the second argument. For $\vec{v}, \vec{w} \in TX$, the expression $\Psi(x+\vec{v}, y+\vec{w}) - \Psi(x+\vec{v}, y) - \Psi(x, y+\vec{w}) + \Psi(x,y) = \delta\Psi(x+\vec{v}, \vec{w}) - \delta\Psi(x,\vec{w})$ is independent of $x, y \in X$. We define the functional $\delta\delta\Psi(\vec{v},\vec{w}) = \Psi(x+\vec{v}, y+\vec{w}) - \Psi(x+\vec{v}, y) - \Psi(x, y+\vec{w}) + \Psi(x,y)$, which turns out to be bilinear on $TX \times TX$.

Definition C.1. *An application $\Phi : X \to \mathbb{R}$ is said to be a quadratic form on X if there exists a bi-affine functional $\Psi : X \times X \to \mathbb{R}$ such that $\Phi(x) = \Psi(x,x)$, for all $x \in X$. It is allways possible to choose a symmetric bi-affine functional Ψ (by eliminating its anti-symmetric part).*

Note that $\delta\Psi(x,\vec{v}) = D\Phi(x)(\vec{v})$ and $\delta\delta\Psi(\vec{v},\vec{w}) = D^2\Phi(x)(\vec{v},\vec{w})$, where $D\Phi$ and $D^2\Phi$ are the first and the second derivatives of Φ.

Proposition C.2. *Let Φ be a quadratic form on the affine space X ($\Phi(x) = \Psi(x,x)$, with Ψ bi-affine symmetric).*

a) Suppose that

$$\delta\delta\Psi(\vec{v},\vec{v}) \geq 0\,, \quad \forall\, \vec{v} \in TX\,.$$

Then, a point $x \in X$ is a global minimum for Φ if and only if

$$\delta\Psi(x,\vec{v}) = 0\,, \quad \forall\, \vec{v} \in TX\,. \tag{C.1}$$

Note that the set of points $x \in X$ satisfying the above property is an affine subspace of X (or the empty set).

b) Suppose that

$$\delta\delta\Psi(\vec{v},\vec{v}) > 0\,, \quad \forall\, \vec{v} \in TX\,,\ \vec{v} \neq 0\,.$$

Then property (C.1) implies that x is a strict global minimum for Φ.

Proof: For any two points $x, y \in X$, by taking $\vec{v}$ the vector pointing from x to y (which can be denoted as $\overrightarrow{xy}$, or as $y - x$), we obtaing through direct computations the equality $\Phi(y) = \Psi(y,y) = \Psi(x,x) + \delta\delta\Psi(\vec{v},\vec{v}) + 2\,\delta\Psi(x,\vec{v})$. If x has property (C.1), we get $\Phi(y) = \Phi(x) + \delta\delta\Psi(\vec{v},\vec{v})$. So, in case a) we obtain that x is a global minimum, while in case b) we deduce that x is a strict global minimum.

As $\delta\Psi$ is affine in its first argument, for each $\vec{v} \in TX$, the set of solutions of $\delta\Psi(x,\vec{v}) = 0$ is an affine subspace of X. Thus, the set of points $x \in X$ with the property (C.1) is an intersection of affine subspaces, which is either the empty set or an affine subspace of X. $\square$

Proposition C.3. *Let A be a $n \times n$ matrix of the form*

$$A = \begin{pmatrix} x_1 & y & \cdots & y \\ y & x_2 & \cdots & y \\ \vdots & \vdots & \ddots & \vdots \\ y & y & \cdots & x_n \end{pmatrix}$$

with $x_k \geq 0 \geq y$, $k = 1, 2, \ldots, n$. If $\det A \geq 0$ then A is semi-positive.

Proof: We shall prove by induction that all principal minors of A are non-negative. In order to simplify the formulas, we shall only show the calculations for $n = 4$. We develop the determinant of A along the first raw.

$$\det \begin{pmatrix} x_1 & y & y & y \\ y & x_2 & y & y \\ y & y & x_3 & y \\ y & y & y & x_4 \end{pmatrix} = x_1 \det \begin{pmatrix} x_2 & y & y \\ y & x_3 & y \\ y & y & x_4 \end{pmatrix} - y \det \begin{pmatrix} y & y & y \\ y & x_3 & y \\ y & y & x_4 \end{pmatrix} -$$

$$- y \det \begin{pmatrix} x_2 & y & y \\ y & y & y \\ y & y & x_4 \end{pmatrix} - y \det \begin{pmatrix} x_2 & y & y \\ y & x_3 & y \\ y & y & y \end{pmatrix} =$$

$$= x_1 \det \begin{pmatrix} x_2 & y & y \\ y & x_3 & y \\ y & y & x_4 \end{pmatrix} - y^2((x_3 - y)(x_4 - y) + (x_2 - y)(x_4 - y) + (x_2 - y)(x_3 - y)) .$$

Taking into account that $x_k \geq 0 \geq y$, one has

$$\det \begin{pmatrix} x_1 & y & y & y \\ y & x_2 & y & y \\ y & y & x_3 & y \\ y & y & y & x_4 \end{pmatrix} \geq 0 \quad \text{implies} \quad \det \begin{pmatrix} x_2 & y & y \\ y & x_3 & y \\ y & y & x_4 \end{pmatrix} \geq 0$$

and the result is proven by applying induction. $\square$

Proposition C.4. *Define the spaces $V = L^2([0, 1])$ and $V_0 = \{v \in V : \int_0^1 v(x)dx = 0\}$. A function $u \in V$ has the property*

$$\int_0^1 u(x)v(x)dx = 0 , \quad \forall\, v \in V_0 \tag{C.2}$$

if and only if u is constant (almost everywhere in $[0, 1]$).

Proof: Consider the linear operator $\mathcal{I} : V \to V$, defined as $\mathcal{I}(v) = \int_0^1 v(x)dx$ (a constant function in $L^2([0, 1])$). It is easy to check that $\mathcal{I}$ is continuous and self-adjoint. The kernel of $\mathcal{I}$ is V_0. Property (C.2) says that u is orthogonal to the kernel of $\mathcal{I}$. By a classical result (see e.g. [30], T.2,XIII,7;2) we get that u belongs to the closure of the image of $\mathcal{I}$. But the image of $\mathcal{I}$ (the set of constant functions) is a closed subspace of $L^2([0,1])$, so u is constant.

It is possible to give a more elementary proof, which makes no use of functional analysis. For any function $u \in L^2([0,1])$, define the set A of all real numbers a such that $u \geq a$ a.e. in $[0,1]$ and the set B of all real numbers b such that $u \leq b$ a.e. in $[0,1]$. It is easy to check that the sets A and B are non-void, A is a semi-interval unbounded below, B is a semi-interval unbounded above and for any two elements $a \in A$, $b \in B$, one has $a \leq b$. Considering a function u satisfying property (C.2), it is sufficient to prove that $\sup A = \inf B$ in order to conclude that u is constant a.e. in $[0,1]$. Suppose $\sup A < \inf B$ and define $c = (\sup A + \inf B)/2$ and $\varepsilon = (\inf B - \sup A)/3$. Then it is not difficult to build two subsets M and N of $[0,1]$ such that $\mathcal{L}_1(M) = \mathcal{L}_1(N) > 0$, $u \leq c - \varepsilon$ on M and $u \geq c + \varepsilon$ on N. Define the function v to be equal to 1 on N, equal to -1 on M and zero outside. This function violates condition (C.2). $\square$

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